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NOTES, PROBLEMS AND LABORATORY EXERCISES IN

Mechanics, Sound, Light, Thermo-Mechanics and Hydraulics

PREPARED
FOR USE IN CONNECTION WITH THE COURSE
IN NATURAL AND EXPERIMENTAL PHILOSOPHY
AT THE UNITED STATES MILITARY ACADEMY

BY

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FIRST EDITION

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Municipal and Structural

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PREFACE

THIS book contains "Notes on Mechanics" from the course at Massachusetts Institute of Technology; "Notes on Graphical Statics," Problems and Notes from the course in the Department of Natural and Experimental Philosophy; and an extensive list of Problems used at the Standard Schools of Technology. It was prepared as a ready reference on the subjects covered by the Department of Natural and Experimental Philosophy, U.S.M.A.

Also a period of approximately one month of practical experimental work has been arranged for. This course is founded on the general line of work at Worcester Polytechnic Institute, and, I believe, will be a useful arrangement. Copies of the exercises to be performed by the students *themselves* in the *laboratory* are contained herein for convenience.

At the Military Academy it has been difficult for the student to refer to a subject under study in any other source than his textbook. This is due largely to lack of time.

By consolidating the subject matter it is believed that, in connection with the course, and later, the information contained in this book may prove of some real service.

While it is intended primarily as a reference book for use with any text on the general subject, portions of the notes and certain of the problems can be included in the regular assignments of lessons where it appears to be advantageous.

I desire to express my gratitude to Colonel Gordon and Colonel Fiebeger, U. S. M. A.; President Hollis, Professors Masius and Haigis, Worcester; Professors Church, Nichols and Richtmyer, Cornell; Professors Miller, Clifford, Drisco, Cross, Derr and Hayward, Massachusetts Institute of Technology; Professors Martin and LePage, Stevens; Professors Williams and Crockett, Rensselaer Polytechnic Institute; Professor Hoskins, Leland Stanford, and Professor Henderson, of Michigan.

In June, 1917, the undersigned received authority from the

War Department to undertake a revision of the course, and it was largely due to the courtesy, help and advice of these gentlemen that I have been able to arrange the notes submitted and to complete the work in the somewhat limited time available.

HALSEY DUNWOODY.

Acting Professor of Natural and Experimental Philosophy.
In Charge of Department.

July 25, 1917.

PUBLISHERS' NOTE

Shortly after this preface was written, Capt. Dunwoody was ordered to military field duty.

J. W. & S.

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NOTES, PROBLEMS
AND LABORATORY EXERCISES

NOTES ON MECHANICS

Used by permission of Prof. C. R. Cross, Mass. Inst. Tech.

Definitions

1. Motion. Motion is change of position in space. We can be acquainted with none but relative motions, as we can know that a body really changes its position only by comparison with some other not possessing the same movement. Our use of the term "rest" is also relative.

In pure motion of translation all the points of a body move with the same velocity and in the same direction at the same instant. When the points of a body describe arcs of concentric circles about some fixed axis, the motion is one of pure rotation. All possible varieties of motion may be produced by the combination of translation and rotation.

2. Velocity. As the term is commonly used, velocity is speed, or rate of motion. It is expressed analytically by the formula $v=\frac{ds}{dt}$. For uniform motion it is evident that speed equals the space traversed in any given time divided by that time.

In physical treatises, however, it is usual to distinguish between velocity and speed, making the former a vector quantity, involving both magnitude and direction, while the latter denotes magnitude only as defined above.

It is furthermore evident that the speed of a particle at any instant is always the space which it would describe in a unit of time were its speed to remain of the same value during that time.

It is customary to express speed in terms of the number of units of length which would be thus traversed in a second; e.g., a freely falling body at the end of its first second of fall has a speed of 9.8 meters per second.

3. Force. We have come to ascribe every change that occurs in the condition or qualities of matter to the action of force. Force is commonly defined as that which causes or tends to cause a change of condition. A mechanical force, or pondero-motive force as it is sometimes called, tends to produce motion of a mass.

An electro-motive force, on the other hand, tends to produce a flow of electricity. A magneto-motive force tends to produce magnetic flux. But these are not forces in a mechanical sense.

It will readily be seen that what we call " force " is really known to us only as a conception. All that we learn directly from experience is that given certain conditions certain results invariably follow. We conceive of something which we call force as acting to bring about these results.

Thus, when a stone is raised above the surface of the earth it tends to fall to the earth, and will do so if unsupported, while if supported it exerts a pressure upon the support. This tendency we ascribe to a force which we call " gravity " acting between the stone and the earth. But all that we know experimentally is that the stone and earth tend to approach each other, and that this tendency has a definite magnitude in any particular case.

Also, when we speak of a force as acting for a certain time, the only fact that we know from observation is that the conditions under which certain actions take place persist for a certain time.

4. Composition of Velocities and Forces. We shall for the present consider only the case in which these are applied at a point.

Velocities and forces are vector quantities. Hence, the combined effect of resultant of a number of velocities or forces can be found by the usual process of vector addition. This is the case whether the components lie in the same or in different planes.

Applications of this principle are the familiar propositions known as Parallelogram and Triangle of Velocities, the Parallelogram and Triangle of Forces, the Polygon of Forces, the Parallelopiped of Forces.

5. Equilibrium of Forces. It follows from the facts stated in the preceding paragraph that two forces are in equilibrium when they are equal and opposite, since their resultant is then equal to zero. Also for a like reason any number of forces are in equilibrium when they can be represented by all the sides of a polygon taken in order.

6. Couple. A moment (*torque*) or a couple (see § 29), is also a vector quantity, since it acts to produce pure rotation in a definite plane. It is customary to represent a moment (*force*×*arm*) by a distance laid off on the axis, right-handed rotations being made positive. Couples may be combined by ordinary vector addition, as will be explained later.

7. Pressure and Impulse. An impulse is to be considered merely as a pressure acting for a very short time. Any pressure may be considered as due to a series of recurring impulses, separated by very short intervals. Such, for example, is the explanation of gaseous pressure offered by the Kinetic Theory.

8. Mass. The term " mass " is commonly used to denote the quantity of matter that a body contains.

9. Measure of Forces. We shall shortly see that all mechanical forces may be compared by their effects in producing or modifying motion.

Steady forces may be measured statically by the use of various forms of dynamometer, of which the ordinary spring balance is an example.

10. Acceleration. Acceleration is rate of change of velocity. It is expressed by the equation $a=\frac{dv}{dt}=\frac{d^2s}{dt^2}$.

It is evident that if the acceleration is constant, that is, if equal increments of velocity are gained in successive equal times, the acceleration is equal to the velocity gained in a unit of time.

Thus the acceleration of a body falling freely under the influence of its weight is 9.8 meters per second per second; often expressed as $\frac{9.8 \text{ m.}}{(\text{second})^2}$. As the unit of time habitually used in physical measurements in the second, this may be more briefly stated as 9.8 m., the expression " per second per second " being understood.

11. Mechanics. Mechanics, in the sense in which the term is ordinarily used, is that branch of physics which treats of the action of force on bodies. It is commonly divided into Statics and Dynamics. Statics treats of balanced forces, or forces in equilibrium; Dynamics, of the action of forces in producing motion. A better nomenclature, however, is that used by Thomson and Tait, as shown in the following quotation from their " Elements of Natural Philosophy " (1873):

" The science which investigates the action of force is called by the most logical writers Dynamics. It is commonly but erroneously called Mechanics; a term employed by Newton in its true sense, the Science of Machines and the Art of making them.

" Force is recognized as acting in two ways: (1) so as to compel rest or to prevent change of motion; and (2) so as to produce or to change motion. Dynamics, therefore, is divided into two parts, which are conveniently called Statics and Kinetics."

Three Laws of Motion

The following propositions, first clearly and collectively stated by Newton, are shown to be true by universal experience. The paragraphs in quotation marks are given in translation as originally stated by Newton in the " Principia," published in 1687.

12. Law I.—" *Every body continues in its state of rest, or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.*"

Another mode of stating the law is the following: *A body at rest continues in that state, and a body in motion proceeds uniformly in a straight line, unless acted upon by some external force.* This truth is the principle of the *Inertia of Matter.*

Illustrations of Law I.: Railroad accidents; phenomena observed by standing passenger on street-railway cars; coursing; fixing head of hammer.

13. Law II. " *Change of motion is proportional to the force applied, and takes place in the direction in which that force acts.*" The term " change of motion," as used by Newton, is identical with the term " change of momentum," now universally employed.

14. Momentum. The product MV of a mass M by its velocity V is called its momentum, which is evidently proportional to both M and V. If we call a the acceleration produced by any force F acting upon a mass M, the rate of change of momentum will be the mass multiplied by the rate of change of its velocity, or Ma. Hence the following statement is true: *Forces are proportional to the momenta which would be generated by their constant and uniform action during a unit of time, or equal times;* or, in other words, *the unbalanced force acting upon a body is measured by the rate of change of momentum thereby produced.* Algebraically this is indicated by the expression $F \propto Ma$, in which F is the force producing an acceleration a in the mass M.

If the force F acts in opposition to a motion already existing in the body, a may be regarded as negative.

If we suppose the same force F to act on different masses M_1, M_2, producing accelerations a_1, a_2, respectively, we shall have $F \propto M_1a_1$, $F \propto M_2a_2$, whence $M_1a_1 = M_2a_2$ and $a_1 : a_2 :: M_2 : M_1$. Hence, *The velocities impressed upon different masses by the action of equal forces* *during a unit of time, or during equal times, are inversely proportional to those masses.*

The following propositions are derived immediately from the relation $F \propto Ma$.

If M is constant, F is proportional to a; whence, *Forces are proportional to the accelerations which they impress upon equal masses.*

If a is constant, F varies as M, whence, *Forces are proportional to the masses upon which they impress equal accelerations.*

It will be shown later that the momentum MV generated by the action of a force F on a mass M for a time t is equal to the magnitude of the force multiplied by its duration; i.e., $Ft = MV$. The quantity Ft is called the *impulse* of the force. Hence the effect of a force is measured by its magnitude multiplied by the time during which it acts.

15. Corollary to Law II. *When any number of forces act simultaneously upon a body, then, whether the body be originally at rest or in motion, each force produces exactly the same effect in magnitude and direction as if it acted alone.* This principle is often called the *Law of the Independence of Motions.*

Illustrations. Man walking or writing on vessel; Peary's sailors on ice-floe; body falling from an elevation; cannon-ball fired in different directions with regard to motion of earth.

16. Law III. "*To every action there is always an equal and contrary reaction*; *or, the mutual actions of two bodies are always equal and oppositely directed.*"

Illustrations. Attraction or repulsion of magnets; action of uncoiling spring; recoil of gun; mutual attraction of earth and mass.

17. Time Required to Produce Motion of a Mass. When a force is applied to a large mass the mass necessarily acquires velocity only with comparative slowness; that is, the acceleration is small. The greater the mass, with a given force, the more slowly does the mass gain in speed. Hence, if the force is applied to the mass, for example, by means of a spring or cord, the more sudden the application of the force, the more will the spring or cord be stretched. The mass seems to offer a resistance to entering into motion, though it is nevertheless acquiring velocity at the rate conditioned by the second Law of Motion; i.e., $a = \frac{F}{M}$.

This effect is often said to be due to the "time required to overcome the inertia of a body." Such a statement is erroneous if the word "inertia" is used in its strict sense; viz., as denoting the absolute inability of matter to change its state except under the action of force. But this term is very commonly used in a somewhat different though analogous sense as denoting that property of matter in virtue of which a definite force is necessary to produce a given change in the existing state of a mass. This last is simply a mode of expressing the general idea of momentum. The same idea is often expressed by the statement that " time is required to produce motion in a mass."

Many peculiar phenomena are explained in virtue of this action; as, for example, the feat of firing a candle through a board, the breaking of the harness under the sudden pull of a horse, the bursting of cannon with high-power explosives, surface-blasting with dynamite, and the like.

The three Laws of Motion, as is the case with all physical principles, are known to be true only from observation and experiment. The best proof that we have of their universality is found in the accordance of observed with predicted results, as, for example, in Astronomy in the case of ellipses, occultations, planetary and cometary motions and other phenomena in which the mutual actions of masses of matter are concerned.

Analytical Statics

18. Analytical Statics. The following elementary applications of the principles of analytical statics are inserted here for convenience of reference.

19. Composition and Resolution of any Number of Forces Applied at a Point. (*a*) Forces in one Plane. Referring to Fig. 1, it will be seen that if AC represents any force R, then AD and AB will represent the rectangular components of this force. Hence, calling these components F_1, F_2, respectively, and denoting by α the angle made by F_2 with R, we have

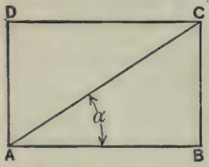

FIG. 1.

$$F_1 = R \sin \alpha, \; F_2 = R \cos \alpha.$$

It will also appear from a consideration of Fig. 1 that if several forces F_1, F_2, F_3, etc., in the same plane and applied at a point, make angles α_1, α_2, α_3, with the axis of X, the sum of their components resolved parallel to X is $F_x = \Sigma F \cos \alpha$, and the sum of their components parallel to Y is $F_y = \Sigma F \sin \alpha$. Since the resultant

$$R = \sqrt{F_y^2 + F_x^2},$$

we have for its value

$$R = \sqrt{(\Sigma F \sin \alpha)^2 + (\Sigma F \cos \alpha)^2}.$$

Also, if θ is the angle made by R with X, $\tan\theta=\frac{\Sigma F\sin\alpha}{\Sigma F\cos\alpha}$.

For equilibrium $\Sigma F\sin\alpha=0$, $\Sigma F\cos\alpha=0$.

(*b*) *Forces in Space.*—For a force in space it is easy to see that if the point of application be taken as the origin of coordinates, the components F_x, F_y, F_z, along the axes are respectively $F_x=F\cos\alpha$, $F_y=F\cos\beta$, $F_z=F\cos\gamma$, α, β, γ being the direction angles of the force F. Also, $F^2=F_x^2+F_y^2+F_z^2$. Hence for a system of forces

$$R=\sqrt{(\Sigma F\cos\alpha)^2+(\Sigma F\cos\beta)^2+\Sigma(F\cos\gamma)^2}.$$

Calling α_r, β_r, γ_r, the direction angles of R, we have

$$\cos\alpha_r=\frac{\Sigma F_x}{R}=\frac{\Sigma F\cos\alpha}{R},$$

$$\cos\beta_r=\frac{\Sigma F_y}{R}=\frac{\Sigma F\cos\beta}{R},$$

$$\cos\gamma_r=\frac{\Sigma F_z}{R}=\frac{\Sigma F\cos\gamma}{R}.$$

The conditions of equilibrium with such a system are

$$\Sigma F\cos\alpha=0,\ \Sigma F\cos\beta=0,\ \Sigma F\cos\gamma=0.$$

20. Center of Parallel Forces. The center of a system of parallel forces is the point through which the resultant always passes, however the direction of the component forces, while still remaining parallel, may be shifted.

It may be found as follows:

Suppose x_r, y_r, z_r, to be the coordinates of the center, x, y, z being those of the point of application of any component F. If we assume the component forces to be made parallel to Y, it follows that $x_r\Sigma F=\Sigma Fx$, since the moment of the resultant $R=\Sigma F$, relatively to the axis Z, must be equal to the sum of the moments of the various components relatively to the same axis, i. e., to ΣFx. In like manner, if the components be made parallel to Z and to X, $y_r\Sigma F=\Sigma Fy$, $z_r\Sigma F=\Sigma Fz$. These equations give the values of x_r, y_r, z_r, and so determine the position of the center.

21. Center of Mass, Center of Gravity, Center of Inertia. These terms all apply to the same point, which is the center of the system of parallel forces formed by the weights of the particles composing the body or system of bodies.

It follows from the last preceding demonstration that for a system of masses the coordinates of the center of mass are

$$x_r=\frac{\Sigma Wx}{\Sigma W}=\frac{\Sigma Mx}{\Sigma M},\ y_r=\frac{\Sigma Wy}{\Sigma W}=\frac{\Sigma My}{\Sigma M},\ z_r=\frac{\Sigma Wz}{\Sigma W}=\frac{\Sigma Mz}{\Sigma M}.$$

For a single body of mass M,

$$x_r=\frac{\int xdM}{\int dM},\quad y_r=\frac{\int ydM}{\int dM},\quad z_r=\frac{\int zdM}{\int dM}.$$

If the body is homogeneous,

$$x_r=\frac{\int xdV}{\int dV},\quad y_r=\frac{\int ydV}{\int dV},\quad z_r=\frac{\int zdV}{\int dV}.$$

The integrals must, of course, be taken between the proper limits.

If the origin is at the center of mass, $x_r=0$, $y_r=0$, $z_r=0$; whence $\int xdM=0$, $\int ydM=0$, $\int zdM=0$.

Particular Cases of Rectilinear Motion

22. Uniform Motion. It appears from the foregoing that a body may remain at rest either when acted upon by no force whatever, or under the action of a system of balanced forces, since in each case the effective force acting is 0. The second case is the only one actually realized.

Also a body may move uniformly in a straight line, either under the action of no force or under the action of a system of balanced forces. In the second case, the only one actually realized, the body is often said to be in dynamical equilibrium.

Illustrations of such a condition of motion are found in the uniform motion of a steamer when the motive force is just equal to the resistance of the water and air; and in the case of a railway train when the motive force is just balanced by frictional and other resistances.

23. Uniformly Variable Motion. A body acted upon by an unbalanced force moves with a variable motion. The most important case is that in which the force is constant and in the direction of the body's motion, in which case the acceleration is constant and the motion is uniformly accelerated.

That the acceleration, $\frac{dv}{dt}=a$, is constant follows immediately from the second Law of Motion, since the effect of a constant force is to produce a constant rate of change of speed.

It is furthermore true that $a=\frac{dv}{dt}=\frac{d^2s}{dt^2}$. Hence by integration between the limits 0 and t we reach the familiar formulæ for uniformly accelerated motion when the body starts from a state of rest under the action of the accelerating force, viz.:

$$v=at \quad (1) \qquad s=\tfrac{1}{2}at^2 \quad (2) \qquad v=\sqrt{2as} \quad (3).$$

Equation (3) is obtained by eliminating t from (1) and (2).

For freely falling bodies we have $v=gt$, $h=\frac{1}{2}gt^2$, $v=\sqrt{2gh}$, where h is the distance fallen through and g the acceleration due to gravity.

If the body possesses an initial velocity v_0 when the accelerating force begins its action, $v_t=v_0+at$, $s_t=v_0t+\frac{1}{2}at^2$, v_t, s_t, being respectively the velocity acquired and space traversed in t seconds.

If the force acts in opposition to an already-acquired motion, the acceleration is negative and the motion is uniformly retarded. In this case $v_t=v_0-at$, $s_t=v_0t-\frac{1}{2}at^2$. Also $s_t=\frac{{v_0}^2}{2a}$.

The final equation assumes that the body is brought to a state of rest.

24. Measure of Impulse. The origin of the equation $Ft=MV$ will now be clear. If a force F acts for a time t on a mass M, it is evident that the resulting change in velocity is $V=at$. But $F=Ma$; whence $Ft=Mat=MV$. That is, the impulse is measured by the resulting change of momentum.

25. Comparison of Forces with Gravity. If a force F acts on a mass of weight W, the acceleration a imparted can be determined from the proportion $F:W::a:g$, whence $a=g\frac{F}{W}$.

26. Motion over Inclined Plane. It will be seen, by a reference to Fig. 2, that for a body of weight W descending a frictionless inclined plane, of height H and length L, the accelerating force will be $F=W\frac{H}{L}$.

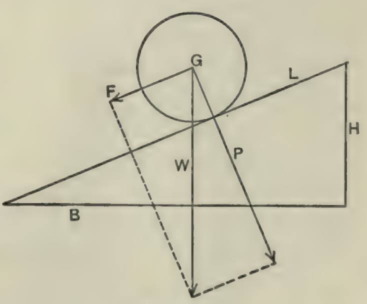

FIG. 2.

Hence, as $F : W :: a : g$, the acceleration of the body will be $a=g\frac{H}{L}$. As a is constant, the motion will be uniformly accelerated. By substituting this value of a in the general formulæ for uniformly accelerated motion, we derive formulæ for the case of a body descending an inclined plane, as follows:

$$v=gt\frac{H}{L},\quad s=\tfrac{1}{2}gt^2\frac{H}{L},\quad v=\sqrt{2gs\frac{H}{L}}.$$

Denoting the angle of slope of the inclined plane by θ, we have $\frac{H}{L}=\sin\theta$ and hence

$$v=gt\sin\theta,\quad s=\tfrac{1}{2}gt^2\sin\theta,\quad v=\sqrt{2gs\sin\theta}.$$

Additional Propositions

27. Resultant Momentum. The resultant momentum in any direction of a system of bodies is $M_r=\Sigma mv$, v being the resolved component of the velocity of m in the direction considered. Thus the resultant momentum of a system of particles parallel to X is $M_x=\Sigma mv_x=\Sigma mv\cos\alpha$, that parallel to Y, $M_y=\Sigma mv_y=\Sigma mv\cos\beta$, that parallel to Z, $\Sigma M_z=\Sigma mv_z=\Sigma mv\cos\gamma$. Also

$$M_r=\sqrt{M_x^2+M_y^2+M_z^2}.$$

28. Relation to Center of Mass. The resultant momentum of any system of bodies is the same as if they were concentrated at the center of mass of the system.

Let m_1, m_2 be the masses of two bodies at distances l_1, l_2 from any assumed coordinate plane, and let l_g be the distance of their center of mass from the same plane. Then, since the moment, relatively to the plane, of the sum of the masses assumed to be concentrated at their center of mass is the same as the sum of their separate moments (see § 21), we have $(m_1+m_2)l_g=m_1l_1+m_2l_2$.

Taking the derivative of each term relatively to time we have $(m_1+m_2)\frac{dl_g}{dt}=m_1\frac{dl_1}{dt}+m_2\frac{dl_2}{dt}$. But the derivatives represent velocities of m_1, m_2, separately and of (m_1+m_2) assumed to be concentrated at their center of mass. Calling these v_1, v_2, v_g, respectively- we have $(m_1+m_2)vg=m_1v_1+m_2v_2$. The second member of the equation represents the resultant momentum of m_1, m_2, and the first member represents the momentum of the masses concentrated as assumed.

The proof is general, for with any number of additional masses whatsoever we may follow the same process, combining them successively. Thus we may apply the same reasoning as before to a third mass m_3 in connection with the masses m_1, m_2 supposed to be concentrated at their center of mass; and so on.

It also follows from what precedes that the resultant momentum of any system of bodies relatively to their center of mass is zero.

29. Couple. If two equal and opposite parallel forces, not acting in the same straight line, are applied to a body, their algebraic sum is zero, and hence there is no tendency toward motion of translation. But as their resultant moment relative to any point can never become zero, their sole effect will be to cause rotation about an axis.

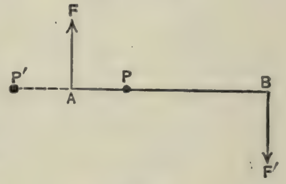

Fig. 3.

The rotary effect, torque or moment of a couple is measured by the product of either force into the length of the arm.

Let F, F' (Fig. 3) constitute a couple whose arm is AB. To find the rotary effect, let P be any point. The moments of F, F', relatively to P, are $F \times AP$, $F' \times BP$. Hence, the total resultant moment of the two forces is $F \times AP + F' \times BP = F \times AB$.

If the point be taken on the same side of both forces, as at P', the resultant moment is $F' \times P'B - F \times P'A = F \times AB$, as before. Hence $T = Fl$, where l is the arm.

A couple may be turned in its own plane or moved parallel to itself without altering its efficiency, since the product Fl remains constant. Also, any couple may be replaced by an equivalent one having a given arm. If l_0 be the arm, the corresponding force will be $F_0 = \frac{T}{l_0}$.

30. Combination of Couples having the Same Axis. *The resultant moment of any number of couples lying in the same or parallel planes is equal to the algebraic sum of their separate moments.*

Calling T_r the resultant moment, $T_r = \Sigma Fl$. For equilibrium, so far as rotation is concerned, $\Sigma Fl = 0$.

Couples not having the same axis may be compounded by a process similar to that used in compounding oblique forces. (See § 94.)

Evidently a couple cannot be balanced by any single force, but only by the application of an equal and opposite couple.

31. Total Effect of Force on Free Body. The tendency of any force acting upon a body is, in general to produce (1) a translatory motion, and (2) a rotation. If the force acts through the center of mass of the body, the resultant moment relatively to that point is zero, and there is no tendency to rotation if the body is free. This will be clear if we consider the force in question to be resolved into an infinite number of parallel forces applied to each particle of the body, and proportional to the mass of that particle. These would tend to cause equal accelerations in each particle, and hence to cause all points in the body to move with the same velocity.

Because of this property of the center of mass it is often called the *center of inertia.*

If the line of action of the force does not pass through the center of mass, the total translatory effect will be the same as if the force were applied at that point, and the rotary effect will be equal to its moment with regard to the center of mass. In Fig. 4 let F be a force applied at A. The condition of the body will not be altered if we imagine two opposite forces, F_1, F_2, each equal and parallel to F, to be applied at G, the center of gravity. But we have now a couple whose moment is $F \times AG$, and a force F_1 equal and parallel to F, and passing through G.

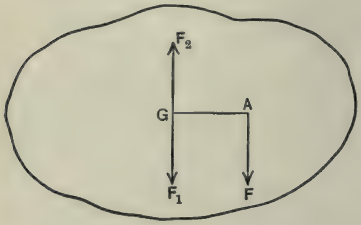

FIG. 4.

It can be shown that the rotation produced by the couple $F \times AG$ will take place about an axis passing through the center of mass of the body.

Every possible motion of a rigid body may be considered as compounded of a rotation about its center of mass combined with a translatory motion.

It follows from what precedes that any number of forces acting upon a body or system of bodies may be replaced by a single resultant force passing through the center of mass, and a resultant couple.

32. Transferability of Force. Since the translatory effect of a force is solely to produce motion in its line of action while the rotary effect is its moment, the total mechanical effect of a force will not be altered by transferring its point of application to any point in its line of action.

This principle facilitates the process of composition of two or more oblique forces not applied at the same point. The lines of action of any two components may be prolonged till they meet in a point, in which case the Parallelogram of Forces becomes directly applicable. The remaining components can be combined successively in the same manner.

Measurement of Force and Mass

33. Units and Standards. As a preliminary to the discussion of this subject it will be necessary to explain briefly the nature of the units of length, mass, and time which are adopted in scientific and industrial measurements.

The unit of length in the metric system is the distance between two transverse lines ruled on the surface of a certain standard platinum-iridium bar (a line standard) adopted as such in 1889 and kept at Paris, which is known as the International prototype standard meter. The standard temperature at which the bar is correct is 0° C. This length is as exactly as possible the length at the same temperature of the original *Mètre des Archives* of 1799, which is an end standard. For scientific purposes, however, the centimeter, $\frac{1}{100}$ part of the meter, has been found to be preferable to the meter for use as a unit.

The unit of length in the British system is the Imperial yard, which is the distance between two transverse lines ruled on gold studs inserted in a certain bronze bar (a line standard), when at a temperature of 62° F. This bar was made the legal standard of Great Britain in 1855 and is kept in the Standards Office, Westminster, London. The foot is habitually employed as a more convenient practical unit than the yard.

The legal yard of the United States is defined (1893) as being of such length that 39.37 of its inches shall equal the length of the meter.

The unit of mass in the metric system is the gram. This is a mass equal to $\frac{1}{1000}$ part of the mass of the International standard prototype kilogram, a platinum-iridium cylinder, adopted as a standard in 1889, and placed together with the International prototype meter in the care of the International Metric Commission at St. Cloud, Paris. This kilogram is an exact copy of the original *Kilogramme des Archives* of 1799.

The British unit of mass is the Imperial avoirdupois pound. This is represented by a certain cylindrical piece of platinum, legalized in 1855 and kept at Westminster.

The legal avoirdupois pound of the United States is a pound of such mass that 2.2046 lb. shall equal 1 kg.

Copies of the prototype standard meter and kilogram (" national prototype standards ") are in the possession of each of the principal nations. Those belonging to the United States are kept at the Office of Weights and Measures, in Washington.

The standard meter is found by comparison to be equal to 39.370113 in. of the British standard yard. The standard kilogram is equal to 15432.3564 g. of the British standard pound. This is the legally adopted ratio in Great Britain.

The unit of time universally employed in both the Metric and British systems is the ordinary sexagesimal second of civil life, which is the $\frac{1}{86400}$ part of the length of the mean solar day.

It was originally intended that the meter should be $\frac{1}{10000000}$ part of a quadrant of the meridian in length. The actual meter, however, is slightly shorter than this. There is, moreover, apparently a small difference in the length of the different meridians according to the longitude. The mean length of a quadrant of the meridian is in fact 10,002,000 m. instead of 10,000,000 m. (Clarke, 1880.)

It was also intended that the gram should be precisely the mass of a cubic centimeter of water at the temperature of its maximum density, 4° C. The actual mass of a cubic centimeter of water at 4° C. is, in fact, 0.999972 g. as determined at the International Metric Bureau (1907).

Inasmuch as there is always a possibility of a slow secular change in the molecular constitution of a metal bar and, consequently, of a minute change in its length, the use of the wave-length of a selected kind of homogeneous light as a standard has been suggested at different times by several persons, first by Lamont (1823). Michelson has recently (1892–93) determined with great accuracy and by a direct method the length of such a wave, produced by incandescent cadmium vapor, in terms of the present prototype standard meter, and, hence, determined the length of the meter in terms of this wave-length.

The meter contains 1553163.6 wave-lengths of a certain component (red) ray of cadmium light.

By repeating this measurement many years hence it will be possible to ascertain whether the standard meter has in any way altered in length. Also, if the present standard and all accurate copies of it were to be destroyed it could still be reproduced at any time from this wave-length.

In view of the possible question as to a sensible variation in the length of the day, other units of time than the second have been proposed, as, for example, the period of vibration of an elastic spring of determined material and dimensions under fixed conditions of temperature, etc., " a perennial spring," to use Kelvin's term. A tuning-fork would be the most suitable instrument for the purpose. But aside from uncertainty as to exact constancy of external conditions, the liability to a secular change of rate of the fork arising from slow internal molecular changes would make the use of such a standard unpractical.

It was suggested by Maxwell that the period of vibration of a homogeneous light-wave emitted by incandescent vapor, that of sodium, for instance, might be used. This period is exact and has been accurately determined in terms of the present second. An objection, however, is its excessive smallness, approximately 510×10^{-12} seconds.

34. Dynamical Measure of Force.—Any mechanical force, whether pressure or impulse, may be measured by means of a system based upon the second Law of Motion.

By making a suitable choice of units we may write $F=Ma$ instead of merely $F\propto Ma$.

The unit of force chosen is that force which produces an acceleration unity in a mass unity. This is indicated algebraically in the equation $F=Ma$, as by making M and a each equal to unity, F also becomes equal to unity.

Since $F=Ma$, it follows also that $M=\frac{F}{a}$, that is, the mass of a body may be measured by the constant ratio between any force and the acceleration which that force will produce when acting on the body.

Hence, if W is the force by which a given body of mass M is drawn toward the earth, that is, if W is the weight of the body, and g the acceleration due to gravity, we have, since g is an acceleration produced by a force W acting upon that body, $M=\frac{W}{g}$.

The unit of mass in all systems will weigh (i.e., be drawn to the earth by a force of) g units of force. For, since $M=\frac{W}{g}$, then, when M equals unity, W must be numerically equal to g; that is, the unit of mass must be drawn to the earth by a force of g units of force. Likewise, the unit of force must be $\frac{1}{g}$ part of the weight of the unit of mass.

It is evident that the equation $M=\frac{W}{g}$ is independent of the particular units of mass, force, and length that we may choose to adopt.

It will be clear from what has been said that either the unit of mass or the unit of force may be selected arbitrarily, and the corresponding unit of force or of mass determined in accordance with the relations which have been shown to exist between them.

Two principal systems of measurement of force and mass have been used, called respectively the *Absolute* and the *Gravitation* System. The former selects the unit of mass and determines the corresponding unit of force; the latter selects the unit of force and determines the corresponding unit of mass.

35. Absolute System. In this system, the *mass* of the gram or pound is chosen as the unit of mass. The mass of a body is therefore, expressed either in grams or in pounds.

The unit of force is determined from this, it being $\frac{1}{g}$ part of the *weight* of the gram or pound. The value of g is commonly expressed in centimeters or in feet, according as metric or British measures are used.

These units were originally proposed by Gauss. It is evident that they are absolutely constant in all places and under all conditions.

The Absolute System is used in all refined physical measurements.

We shall see later that units of measurement of all physical forces can be derived from the fundamental units of length, mass, and time. Such units are called *absolute units*, or, more logically, *derived units.*

The particular form of absolute system now universally employed in science is the *Centimeter-gram-second System* (*C. G. S. System*), so called from the fundamental units of length, mass, and time on which it is based. The *C. G. S.* unit of force is called a *dyne*, and is $\frac{1}{g}$ of the weight of a gram, g being expressed in centimeters. The British *Foot-pound-second System* is based upon the foot, pound, and second. The corresponding unit of force is called a *poundal*, and is $\frac{1}{g}$ of the weight of a pound, g being expressed in feet. This system, however, is practically obsolete.

Various other forms of the Absolute System have been used in past times; as, for example, a *meter-gram-second system*. a *millimeter-gram-second system*, and a *foot-grain-second system.*

It will appear on consideration that the absolute unit of force thus derived will fulfill the requirements of the general definition of a unit of force, and generate an acceleration unity in a mass unity. Thus, for example, the dyne must generate an acceleration of 1 cm. in a mass of 1 g. For if such a mass falls freely, it will acquire an acceleration of g cm. But the accelerating force is g dynes, that being the weight of a gram. Hence, under the action of a force of 1 dyne the acceleration acquired would be 1 cm.

As g has a value of 980.9 cm. at Paris and the absolute unit of force is equal to $\frac{1}{g}$ of the weight of the unit of mass *at any place whatever*, the dyne is equal to $\frac{1}{980.9}$ of the weight (force of attraction to the earth) of a gram *at Paris*. And as the value of g at London is 32.191 ft., the poundal is equal to $\frac{1}{32.191}$ of the weight of a pound at *London*.

36. Gravitation System. In this system the *weight* of the pound or gram (that is, the force by which the pound-mass or gram-mass is drawn toward the earth) is chosen as the unit of force. The weight of the kilogram is also frequently used as a unit.

The mass of a body is, therefore, measured by its weight *in pounds* or *grams* or *kilograms*, divided by the acceleration produced by gravity. This quotient is evidently a constant for any particular body; for, if its weight W (that is, the force by which it is attracted to the earth) varies from any cause (as from change of latitude, change of altitude, etc.), the acceleration g will vary proportionally. (Law II.)

Since in all systems the unit of mass weighs g units of force, it follows that the unit of mass in the Gravitation System must weigh g lb., g., or kg., according to which of these is employed as a unit of force. That is, its mass must be g times that of the pound-weight, gram-weight, or kilogram-weight.

In the Gravitation System g is usually expressed in feet or in meters.

The unit of mass in the Gravitation System has never acquired any distinct name, although the term *Slug* or grav has been suggested. When this system is used, care must be taken to express the mass of the body in terms of this unit and not in pounds or grams.

We shall see hereafter that the weight of the pound or gram-mass, and consequently g, vary with the locality, so that the units of both force and mass in the Gravitation System are variable unless the definite value of g at some particular place is assumed as a standard, which has not been customary. This want of definiteness constitutes a fatal objection to the use of the Gravitation System, where the greatest accuracy is required. It is commonly used, however, in ordinary engineering computations in which the variations of g may be neglected.

The value for g of 32.2 feet or 9.8 meters may always be employed in the Gravitation System, no closer approximation being necessary.

The expression for a force in gravitation units may be transformed into absolute units by multiplying the numerical value of the force in gravitation units by the value of g for the locality at which the measurements are made. Methods of determining the value of g for any place will be explained later.

The student will notice that in the discussion of this subject ambiguity is likely to arise from the fact that the terms " pound " and " gram " are used in two different senses, both as denoting a mass, and as denoting the force by which this mass is drawn toward the earth. To remove this ambiguity as far as possible, the term " weight of a pound or gram " has been used in the preceding discussion whenever a force is referred to. The pound and gram are primarily standards of mass, and the use of the same names to denote forces is a secondary application. Whenever the terms " pound " or " gram " are used as denoting forces, they are to be understood as really meaning the *weight* of the units of mass denoted by those names.

In the equation $W = Mg (F = Ma)$, which is true for all systems of measurement of mass and force, W expresses in units of force the downward tendency (due to its weight) of a body of mass M. W and M must always be expressed in units of the same system, either Gravitation or Absolute. Also in both systems g expresses the number of units of force by which a unit mass is drawn to the earth.

A system is sometimes used in ordinary industrial measurements in which the pound is taken as the unit of mass and its weight as the unit of force. In this case the unit of force is implicitly assumed to be the force which produces an acceleration of g feet in the unit of mass and the unit of acceleration to be g ft. rather than 1 ft. This system is devoid of exactness to the same extent as the ordinary Gravitation System.

37. Example. The application of the absolute system may be made clearer by a simple example.

Let us suppose that a certain force A as measured in Paris is equal to the weight of 100 g. at that place. A certain other force B as measured at London is equal to the weight of 1000 g. at that place. Since the force by which a gram is attracted to the earth is not the same at these different localities, we cannot compare the forces directly. If, however, we reduce the forces to absolute units, the difficulty is avoided. The force A expressed in absolute *C. G. S.* units is $A=100\times 980.9=98{,}090$ dynes, and also $B=1000\times 981.2=981{,}200$ dynes; since the value of g for Paris is 980.9 cm., and that for London is 981.2 cm. These results immediately show the absolute relation of the forces A and B.

It is evident that an absolute system may be based upon any convenient units of length, mass, and time. Also a system of measurement might be used in which the unit of force was taken as the invariable weight of the standard pound or gram at some particular place.

A table of the units most commonly used will be found in par. 41.

38. Note. A philosophical study of the nature of force and of our knowledge of it, shows that ultimately we have no way of measuring the relative mechanical magnitude of all kinds of forces except by their effect in generating momentum. Hence, Law II of Motion is rather a definition of a system of measurement than a law determined from measurements obtained by other means. We call forces mechanically equal when they produce an equal rate of change of momentum; and, if one force produces a rate of change n times as great as that produced by a second force, we call the former force n times as great as the latter.

TABLE OF DIMENSIONS

Quantities		*Dimensions*
Length		L
Mass		M
Time		T
Area		L^2
Volume		L^3
Angle	$=\frac{\text{Arc}}{\text{Radius}}$	L°
Velocity	$=\frac{S}{T}$	LT^{-1}
Angular velocity	$=\frac{\theta}{T}$	T^{-1}
Acceleration	$=\frac{V}{T}$	LT^{-2}
Momentum	$=MV$	MLT^{-1}
Force	$=Ma$	MLT^{-2}
Torque	$=Fl$	ML^2T^{-2}
Energy	$=\frac{1}{2}MV^2$	ML^2T^{-2}
Power	$=\frac{1}{2}\frac{MV^2}{T}$	ML^2T^{-3}
Moment of inertia	$=Mk^2$	ML^2

Energy

39. Work and Energy. Work is performed whenever a force produces motion in opposition to a resistance; or more generally whenever a force acts to move its point of application through space.

Work is found to be directly proportional to each of these variables, force and distance. Hence, representing the force by F and the distance through which its point of application is moved by S, the work done is expressed by the equation $E=FS$.

" Energy is the capacity of doing work."—MAXWELL.

If the motion of the point of application is not in the line of action of the force, the component of the motion *in the direction of the force* must be taken.

40. Units. The unit of work in practical use by engineers is the *kilogrammeter* in Metric, and the *foot-pound* in British measures. The kilogrammeter is the work done in overcoming a force equal to the weight of 1 kg. through the space of 1 m. The foot-pound is the work done in overcoming a force equal to the weight of 1 lb. through the space of 1 ft. The kilogrammeter is equal to 7.233 ft.-lb. The *gram-centimeter* is frequently employed as a smaller unit of work. The *foot-ton* is often used when great amounts of energy are to be considered.

In purely scientific investigations, absolute units are commonly employed. The absolute unit of work is the work done in overcoming an absolute unit of force through a unit of length.

The unit of work in the centimeter-gram-second system is called the *erg*, and is the work done in overcoming a force of 1 dyne through 1 cm. It is a *dyne-centimeter*. In the foot-pound-second system, the unit is the *foot-poundal*, which is the work done in overcoming a force of 1 poundal through 1 ft.

The rate of work or work-rate is the work done in a unit of time. That is, $R=\frac{FS}{T}$. If the rate is not uniform, this definition applies to its average value. In general, $R=\frac{dE}{dt}$.

Lord Kelvin has given the title " activity " to the rate of doing work. It is usually called " power " by engineers.

A standard in ordinary practical use, for comparing the rate of work or the activity of different motors, is the *horse-power*, which is equal to 33,000 ft.-lb. per minute, or 550 ft.-lb. (76.0 kilogrammeters) *per second.* One horse-power is equal to 7.46×10^9, or 7,460,000 ergs per second, assuming $g=981$ cm. The French *force de cheval* or *chevalvapeur,* is not identical with the English *horse-power,* but is defined as 75 kilogrammeters per second. This equals 7.36×10^9 ergs per second, if $g = 981$ cm.

A unit of work frequently used in connection with electrical measurements is the *Joule* = 10,000,000 (10^7) ergs. The corresponding unit of activity is the *Watt* = 10^7 ergs per second. The *kilowatt,* a unit commonly used in rating electric machinery, is 1000 watts = 1.34 horse-power.

Engineers often use the *horse-power-hour,* and *watt-hour,* the *kilowatt-hour,* respectively as practical units of work. A horse-power-hour is the work done in one hour when energy is expended at a constant rate of 1 horse-power. The other units referred to are defined in like manner.

The *horse-power* measured in electrical units = 746 watts. The French *force de cheval* is often called the *metric horse-power.* The German *pferdekraft* has the same value, 75 kilogrammeters per second or 736 watts.

In ballistic problems the ton is taken as 2240 lb.

41. Accumulated Work. Kinetic Energy. Frequently the space S in the formula for work done is not directly known, while the velocity with which a body is moving is given. An expression for the work which can be done by a body of mass M, moving with a velocity V, may be found by ascertaining the space through which the body would move against a constant resistance and with a uniformly retarded motion before its velocity would be reduced to zero, and substituting this space as expressed in terms of V in the general equation.

If the body is moving with a velocity V against a resistance F which is capable of producing a retardation a, the distance over which the body will move is $S = \frac{V^2}{2a}$. Substituting this value, we have $E = FS = F\frac{V^2}{2a}$.

But

$$\frac{F}{a} = \frac{W}{g},$$

whence

$$E = \frac{W}{g}\frac{V^2}{2} = \tfrac{1}{2}MV^2.$$

Hence *the work which a moving body is capable of performing in virtue of its motion is equal to half its mass into the square of its velocity.*

The result is expressed in units of either the absolute or the gravitation system, according as the mass is expressed in the one or the other of these.

It will be observed that in the preceding demonstration it is assumed that the work necessary to bring a body to rest is the same, whatever may be the nature of the resistance overcome, which is an experimentally determined fact.

The energy possessed by a body because of its motion is called Kinetic Energy.

The product MV^2, which is twice the kinetic energy of a moving mass, was called by Leibnitz the *vis viva* of the mass.

TABLE OF UNITS

System	Mass	Length	Time	Force	Energy or Work	Power or Activity
Absolute Metric C. G. S.	Gram	Centimeter	Second	Dyne	Erg = Dyne-centimeter	$\frac{\text{Erg}}{\text{Second}}$
					Joule = 10^7 Ergs	Watt $=\frac{\text{Joule}}{\text{Second}}$ = Volt-ampere $=10^7\frac{\text{Erg}}{\text{Second}}$
					Watt-hour Kilowatt-hour	Kilowatt = 1000 Watts
Gravitation Metric	"Metric Slug" or "Grav." weighing 9.8 kg.	Meter	Second	Kilogram	Kilogrammeter	Force de Cheval $=75\frac{\text{Kilogrammeter}}{\text{Second}}$
Gravitation British	"Slug" or "Grav." weighing 32.2 lb.	Foot	Second	Pound	Foot-pound Horse-power-hour	Horse-power = $550\frac{\text{Foot-pound}}{\text{Second}}$

42. Rotating Bodies. It remains to be shown how the kinetic energy of a rigid rotating body is determined. The *angular velocity* of such a body is measured by the angle in circular measure through which it rotates in a unit of time, and is numerically equal to the linear velocity of any point at a distance unity from the axis of rotation. That is, the angular velocity $\omega=\frac{\theta}{t}$ or in general $\omega=\frac{d\theta}{dt}$, where θ is the angle described in t seconds.

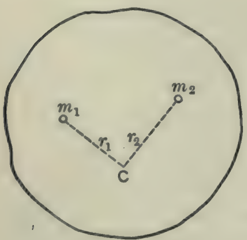

FIG. 5.

Let m_1 be the mass of a particle of the body situated at a distance r_1 from the axis of rotation through C (Fig. 5), and let ω be the angular velocity of the body. The energy of the particle m_1 is $\frac{1}{2}m_1v_1^2$, or $\frac{1}{2}m_1r_1^2\omega^2$, since $v_1 = r_1\omega$.

The energy of any other particle of mass m_2 at a distance r_2 from C is $\frac{1}{2}m_2r_2^2\omega^2$. Similar expressions may be obtained for all the particles of the body. Hence, the total energy of rotation of the body is

$$E = \tfrac{1}{2}m_1r_1^2\omega^2 + \tfrac{1}{2}m_2r_2^2\omega^2 + \tfrac{1}{2}m_3r_3^2\omega^2 \;.\;.\;.\; + \tfrac{1}{2}m_nr_n\omega^2 = \tfrac{1}{2}\omega^2\Sigma mr^2;$$

whence the formula $E = \frac{1}{2}\omega^2\Sigma mr^2$ represents the accumulated energy of the body due to its rotation about C, and is the amount of work which it will perform while being brought to rest.

43. Moment of Inertia. The expression Σmr^2 is of very frequent occurrence in dynamics, and is known as the *moment of inertia* of the mass. Denoting it by I, we have $E = \frac{1}{2}\omega^2 I$. That is, *the energy of rotation of a body is equal to its moment of inertia multiplied by half the square of its angular velocity.* The quantity I evidently varies with the form of the body, its mass, and the position of its axis of rotation.

Using the notation of the Calculus we may write $I = \int r^2dm$.

44. Radius of Gyration. If we assume the total mass of a body to be concentrated in a single point situated at a distance k from the axis of rotation such that the moment of inertia of the mass thus concentrated shall equal the moment of inertia of the distributed mass of the body, then

$$Mk^2 = I \quad \text{and} \quad k = \sqrt{\frac{I}{M}}.$$

The radius k is called the *radius of gyration* of the mass.

45. Moment of Inertia about any Axis. Knowing the moment of inertia I_g with reference to an axis passing through the center of gravity G of a mass it is often necessary to find the moment of inertia I_a with reference to an axis parallel to the former and passing through a point A at a given distance d from G.

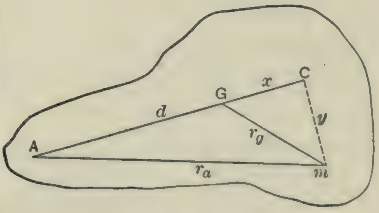

FIG. 6.

In Fig. 6 let m be any element of mass, i_g its moment of inertia relatively to an axis G, i_a its moment of inertia relatively to an axis through A, and denote by r_g and r_a the respective distances of m from G and A. Let $GC = x$, $mC = y$. Then $i_g = mr_g^2 = m(x^2 + y^2)$. Also $i_a = mr_a^2 = m[(x+d)^2 + y^2]$ whence $i_a - i_g = m(2dx + d^2) = 2dmx$

$+md^2$. For any other element m', $i'_a - i'_g = (2dm'x' + m'd^2)$, and for the whole mass $\Sigma i_a - \Sigma i_g = 2d\Sigma mx + d^2\Sigma m$. But $\Sigma mx = 0$, since this is the sum of the values of mx for all the particles composing the body with reference to an axis through the center of gravity. Hence $I_a - I_g = Md^2$ and $I_a = I_g + Md^2$.

46. Determination of Angular Velocity. The angular velocity generated by the expenditure of an amount of work FS upon a body can be found as follows:

$$FS = \tfrac{1}{2}\omega^2 I;$$

whence

$$\omega = \sqrt{\frac{2FS}{I}}.$$

It will be seen also that by solving this equation relatively to I an expression may be found for the moment of inertia, thus indicating a way in which the value of this can be determined experimentally.

Table of Moments of Inertia

From Routh's Rigid Dynamics

The moment of inertia of (1) A **rectangle** whose sides are $2a$ and $2b$, about an axis through its center in its plane perpendicular to the side $2a$, is;

$$\text{mass} \times \frac{a^2}{3},$$

about an axis through its center perpendicular to its plane is;

$$\text{mass} \times \frac{a^2 + b^2}{3}.$$

(2) An **ellipse,** semi-axes a and b, about the major axis a, is;

$$\text{mass} \times \frac{b^2}{4},$$

about the minor axis b,

$$\text{mass} \times \frac{a^2}{4},$$

about an axis perpendicular to its plane through the center is;

$$\text{mass} \times \frac{a^2 + b^2}{4}.$$

In the particular case of a circle of radius a, the moment of inertia about a diameter $= \text{mass} \times \frac{a^2}{4}$, and that about perpendicular to its plane through the center $= \text{mass} \times \frac{a^2}{2}$.

(3) An **ellipsoid,** semi-axes, *a, b, c,* about the axis *a*, is,

$$\text{mass} \times \frac{b^2+c^2}{5}.$$

In the particular case of a sphere of radius a the moment of inertia about a diameter is,

$$\text{mass} \times \frac{2}{5}a^2.$$

(4) A **right solid** whose sides are $2a$, $2b$, $2c$, about an axis through its center perpendicular to the plane containing the sides b and c, is,

$$\text{mass} \times \frac{b^2 \times c^2}{3}.$$

These results may all be included in one rule, as an assistance to the memory.

Moment of inertia about an axis of symmetry is,

$$\text{mass} \times \frac{\text{(sum of squares of perpendicular semi-axes)}}{3,\ 4 \text{ or } 5}.$$

The denominator is to be 3, 4, or 5, according as the body is rectangular, elliptical or ellipsoidal.

It follows from the demonstration given later in § 48, that the total work done upon the particles of a rotating mass must be equal to the kinetic energy due to their linear velocities acquired thereby. That is, $FS = \Sigma \frac{1}{2}mv^2 = \frac{1}{2}\omega^2 I$.

47. Kinetic Energy. Energy is either *Kinetic* or *Potential.* Kinetic energy, as already explained (see § 41), is the capacity possessed by a body of doing work in virtue of its motion, and is expressed by equation

$$K = \tfrac{1}{2}MV^2.$$

Illustrations.—Projectile in motion; fly-wheel; stream of flowing water actuating undershot water wheel; current of air moving windmill.

The term "kinetic energy" is due to Kelvin. It is sometimes called "actual energy," as proposed by Rankine.

From the equation $K = \frac{1}{2}MV^2$ it follows that $M = \dfrac{2K}{V^2}$ from which we can ascertain the mass of a moving body when its kinetic energy and velocity are known. Such a procedure has been employed to determine the mass of an *electron.*

48. Effect of Unbalanced Force of Mass. If an unbalanced force acts to accelerate the motion of a mass, its effect is to add to the kinetic energy of the mass. The increment of kinetic energy, that is, the work stored up, in a given time, is

$$K=\tfrac{1}{2}M(V_1^2-V_0^2).$$

If the mass, being in motion, moves against a resistance, it does work which is measured by the decrease of kinetic energy; that is,

$$K=\tfrac{1}{2}M(V_0^2-V_1^2).$$

The increase of kinetic energy produced by the continued action of a force on a mass is equivalent to the force multiplied by the space through which it acts. This will appear from the following demonstration:

Consider the action of the constant force F on a mass M through a space S. Then as $F=Ma$ and $S=V_0t+\frac{1}{2}at^2$, we have $FS=MV_0at+Ma^2t^2$. This is equal to the gain of kinetic energy in passing through S. For calling K_0 and K_1 the kinetic energy at the beginning and end of S, respectively, $K_0=\frac{1}{2}MV_0^2$, $K_1=\frac{1}{2}M(V_0+at)^2$ whence the gain in kinetic energy, $K_1-K_0=MV_0at+\frac{1}{2}Ma^2t^2$ which has already been shown to be the value of FS. Hence

$$FS=\tfrac{1}{2}M(V_1^2-V_0^2).$$

If the force F is not constant during the time t, we may divide into a very great number of parts so small that during each one of them the force acting may be considered as constant. Then, for each one of the elementary spaces traversed in an element of time, the proposition will hold. Hence we have for their sum in all cases

$$\int_{s_0}^{s_1} FdS=\tfrac{1}{2}M(V_1^2-V_0^2.)$$

This equation may be derived directly by the methods of the Calculus as follows:

$$FdS=Fvdt=Fatdt=(Ma)atdt.$$

Hence

$$\int_{s_0}^{s_1} Fds=\int_{t_0}^{t_1} Ma^2tdt=\left[\tfrac{1}{2}Ma^2t^2\right]_{t_0}^{t_1}=\left[\tfrac{1}{2}MV^2\right]_{V_0}^{V_1}.$$

Hence

$$F(S_1-S_0)=\tfrac{1}{2}M(V_1^2-V_0^2).$$

The demonstration evidently applies whether the force F is constant or variable.

From a similar mode of reasoning, it follows that when a moving mass M overcomes a resistance F through a space S, the work done by the body results in a loss of kinetic energy as expressed by the equation

$$FS=\tfrac{1}{2}M(V_0^2-V_1^2).$$

In the first case considered FS is commonly spoken of as the *work done upon the body* by the force F in time T. In the second case *the work is done by the body* against the force F.

49. Total Kinetic Energy.—The total kinetic mechanical energy of a body is the sum of the energy due to translation and that due to rotation; i.e.,

$$K_t=\tfrac{1}{2}MV^2+\tfrac{1}{2}\omega^2 I.$$

For a system of bodies the sum of the energies of each part must be taken.

V is the linear velocity of the center of gravity of the mass.

50. Equivalent Mass. It will be seen that if a force acts on a mass to produce simultaneous translation and rotation, as, for example, when a ball is caused to roll on a surface, the relation between the work done upon the body and the total resulting kinetic energy is represented by the equation $FS=\tfrac{1}{2}MV^2+\tfrac{1}{2}\omega^2 I$.

If we denote by M_e a mass such that with the linear velocity V alone it would possess an amount of energy equal to the total energy of M, we have $\tfrac{1}{2}M_eV^2=\tfrac{1}{2}MV^2+\tfrac{1}{2}\omega^2 I$. As $V=\omega R$, $M_e=M+\frac{I}{R^2}$.

Thus a sphere descending an inclined plane moves with a linear acceleration only $\tfrac{5}{7}$ *of that which it would acquire did it not rotate, since for it* $M_e=\tfrac{7}{5}M$.

51. Potential Energy. This is the capacity of doing work possessed by the bodies or particles composing a system in virtue of their relative position. If bodies are so situated that they are acted upon by a force which will produce motion in them on the removal of some restraining force, and thence generate kinetic energy, the system is said to have potential energy. Thus a mass suspended at an elevation will fall as soon as the cord sustaining it is cut. The potential energy in this case is $P=FS$, where F is the weight of the raised mass, and S is its elevation; or $P=\tfrac{1}{2}MV^2$ where V is the velocity corresponding to a fall from the height S.

Additional Examples.—Stretched spring; head of water.

In general, the potential energy of a system is measured by the work expended in reducing its energy to zero; or, conversely, by the work which must be done to endow the system with the amount of potential energy which it possesses.

Potential energy is sometimes called "energy of configuration or strain." The term "potential energy" was originally suggested by Rankine.

52. Transformation of Potential Energy into Kinetic and the Reverse.

Examples.—Pendulum; tuning-fork in vibration; hyrdaulic ram.

53. Conservation of Energy. *The sum of the kinetic and potential energies of a system of bodies not acted upon by any external force is a constant.* Or, in other words, *the total energy of a system of bodies is not affected by their mutual actions.*

This fact is learned from observation and experiment, and is found to be true without exception.

As an illustration consider the case of a body falling from a height h_0 and which has descended to a height h_1. The potential energy is Wh_1 and the kinetic energy is $\frac{1}{2}MV_1^2 = W(h_0 - h_1)$. The sum of these is evidently a constant and equal to the original potential energy Wh_0, or to the final kinetic energy on reaching the earth.

The same reasoning can be applied to the case of a body thrown upward, a pendulum, a vibrating tuning-fork or string, etc.

54. Conservative System. "When the nature of a material system is such that if, after the system has undergone any series of changes, it is brought back in any manner to its original state, the whole work done by external agents on the system is equal to the whole work done by the system in overcoming external force, the system is called a *conservative system.*"—MAXWELL.

55. Different Forms of Energy. There are various forms of energy other than mechanical energy, the only kind which we have thus far considered. Several schemes of classification of these have been suggested. The following is a modification of one given long since by Professor Balfour Stewart:

I. VISIBLE MECHANICAL ENERGY.
  a. Visible kinetic energy.
  b. Potential energy of visible arrangement.

II. INVISIBLE MOLECULAR ENERGY.
  a. Kinetic energy of absorbed heat.
  b. Radiant energy (kinetic).
  c. Potential energy due to molecular separation by heat.
  d. Potential energy due to chemical separation.

III. ENERGY OF ELECTRICITY.
  a. Potential energy due to electrical charge.
  b. Kinetic energy due to electrical current.

Another mode of classification possessing certain advantages is the following, given by Professor A. A. Noyes:

1. Kinetic Energy
2. Gravitation Energy
3. Cohesion Energy
4. Disgregation Energy
5. Electrical Energy
6. Magnetic Energy
7. Chemical Energy
8. Heat Energy

56. Note. It is possible that all energy is in fact kinetic, that potential energy, so called, is always due to undetected motion of some kind, though of an unknown nature. While this has not as yet been proved to be the case, it is a plausible hypothesis.

For example, Kelvin has shown that the phenomena of elasticity and hence potential energy of strain may be accounted for by an assumed rotation of elementary portions of a mass. Again, the explanation of gravitation by Le Sage assumes the bombardment of masses by "ultra-mundane corpuscles."

A familiar example of a case in which potential energy possessed by a mass is in fact due to motion is found in the case of gas, which is capable of doing work in virtue of its expansive tendency or pressure. But, according to the Kinetic Theory, this pressure is caused by the rapid motion of the molecules of the gas. The potential energy of the gas is really the kinetic energy of its molecules.

57. Energy Transformations. *Any one of the various forms of energy may give rise, either directly or indirectly, to any other form.*

Illustrations. Heat produced by percussion or friction; steam-engine; electrification by friction; attraction and repulsion of charged bodies: Dynamo-electric generator; electric motor: Thermo-electricity; wire heated by electricity: Heat of chemical combination; dissociation by heat: Voltaic battery; electro-chemical decomposition.

The above proposition relates only to the qualitative aspect of the different transformations of energy. It was formerly known as the principle of the "Correlation of Forces," a term introduced by Grove in 1843, at which early date the distinction between force and energy was not clearly recognized.

58. Conservation of Energy. The following law, which is a statement of the doctrine of the conservation of energy, holds for all known forms of physical energy: "*The total energy of any body or system of* *bodies is a quantity which can neither be increased nor diminished by any mutual action of these bodies, though it may be transformed into any of the forms of which energy is susceptible."*—MAXWELL.

59. Illustrations of Principle. For example, suppose a dynamo-machine to generate a current of electricity which, in its turn, drives an electric motor. If the work done in driving the dynamo is measured, and also the heat generated mechanically by friction of machinery, resistance of air to moving parts, etc., the total heat generated by the current in the complete circuit, and the total mechanical work done by the motor, then the mechanical work done in driving the generator will equal the total amount of energy developed in the several resulting operations.

Again, if a definite weight of zinc in a battery cell is oxidized and the chemical energy disappearing in the cell is wholly transformed into the electrical energy of the current produced, and this current expends its energy in heating a resistance coil, it will be found that the same amount of heat is liberated in the circuit that would have resulted from the direct oxidation of the same weight of zinc in a calorimeter.

60. Case of Heat. When a change of state occurs on raising the temperature of a body, there is always an absorption or liberation of heat, corresponding to the expenditure of work (positive or negative) accompanying the change. Thus, when a liquid is caused to assume a vaporous condition, a great amount of heat disappears (*latent heat of vaporization*). This is due to the fact that mechanical work is done by the expenditure of heat, first in separating the particles of the liquid from one another in the act of evaporation or of boiling (*internal work*), and second, in overcoming the atmospheric pressure which resists the expansion which accompanies the change of state (*external work*). When ice is melted, the large amount of heat that disappears (*latent heat of fusion*) is chiefly expended in doing internal work. Increase of either the internal or external work tends to increase the latent heat. Increased external pressure raises the fusing-point of substances which expand on melting because of the resulting increase in the external work which must be performed. An opposite effect occurs with substances which like ice contract on melting. Anything which increases the internal work accompanying a change of state, as, for example, in vaporization, raises the boiling-point. Thus, we may suppose that the rise of the boiling-point of liquids when solids are dissolved in them is due to the increased difficulty of separating the liquid from the dissolved solid.

In ordinary evaporation the heat which necessarily disappears on account of the change of state is withdrawn from neighboring bodies, thus lowering the temperature to a greater extent according as the evaporation is more rapid.

61. Mechanical Equivalent of Heat. The quantity of mechanical work which is capable of generating one unit of heat, is called the *mechanical equivalent of heat*, or *Joule's equivalent*, from the physicist who first determined its value.

The thermal unit commonly employed by engineers in Great Britain and the United States is the amount of heat necessary to raise 1 lb. of water through 1° F. Owing to the variation of the specific heat of water with temperature the temperature of the water must be stated. The C. G. S. unit of heat is the *gram-degree*, the temperature being measured in Centigrade degrees. The readings are ordinarily reduced to the standard hydrogen thermometer.

Several determinations later than that of Joule have been made by different persons using various methods. The value obtained by Rowland (Baltimore, 1879) with certain corrections subsequently applied, for the temperature 15° C. = 59° F. is $J = 778$ ft.-lb. at Greenwich (B. T. U.). In C. G. S. units $J = 4.19 \times 10^7$ (41,900,000) ergs.

The term *gram-calorie* has come to be generally used instead of *gram degree.*

The student will observe that the two values of J, here given, are not numerically identical, since the unit of heat in the former case (B. T. U.) is the *pound-degree* (Fahrenheit), and in the latter the *gram-degree* (Centigrade).

62. Energy Factors. It is found that with all forms of energy two factors, upon both of which the quantity of energy depends, must be taken into consideration. One of these is called the *capacity factor*, the other the *intensity factor.* The energy is found to depend upon the product of these.

For example, in the expression for mechanical kinetic energy, $K = \frac{1}{2}MV^2$, M is the capacity factor, V^2 the intensity factor. In the case of the energy of a compressed gas the volume is the capacity factor, the pressure the intensity factor. In the case of heat energy, the heat-capacity at a certain temperature is the capacity factor, the temperature the intensity factor. In the case of an electric charge, the electrical quantity is the capacity factor, its potential the intensity factor.

It is the intensity factor which determines the tendency towards change of energy either as to distribution or form. It is because of the pressure of compressed gas that it tends to expand. The temperature of a mass determines the tendency towards a transfer of heat; the potential of an electric charge determines the tendency toward a flow of electricity.

The intensity factor is a relative quantity. Thus, a gaseous pressure, temperature, electric potential are magnitudes relative to some value taken as a standard. Hence, neither transfer nor transformation of energy can take place unless there exists a difference in the intensity factor between different parts of the system. Thus, there is no transfer of gas between two connecting reservoirs if the pressure of the gas is the same in each; there is no resultant transfer of heat between two bodies whose temperature is the same; nor of electricity between two bodies at the same potential.

63. Function of Machines. It will be seen from what precedes that the function of a machine is merely to transfer or transform energy, not to create it.

Neglecting friction and other harmful resistances, the effective force applied to any machine multiplied into the distance through which its point of application is moved in the direction in which the force acts, is equal to the resistance overcome multiplied into the distance through which it is overcome. That is, the mechanical work supplied to the machine is equal to the mechanical work done by the machine in overcoming resistance.

On account of friction and various other harmful resistances, the latter of these is in practice always less than the former.

Examples. Lever, wheel and axle, pulley, and other "mechanical powers."

The *efficiency* of a machine is the ratio of the work which it does to the work done upon it.

64. Measurement of Work of Machinery. The mechanical energy expended in driving a machine may be measured by different forms of dynamometer or erg-meter.

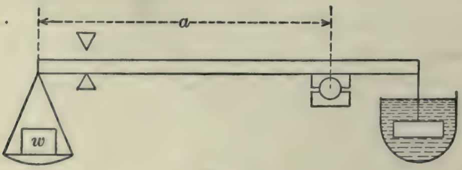

FIG. 7.

1. *Friction Brake.* The shaft (Fig. 7) is made to run at its normal speed, overcoming the friction of the brake as a resistance. Calling this speed n (R.P.M.), w the balancing weight in the brake-pan, a the arm of the brake, the work done by the machine is $W = wa \times 2\pi n$

per minute, whence $\text{H.P.} = \frac{2\pi nwa}{33{,}000}$, if w and a are expressed in pounds and feet respectively.

2. *Transmission Dynamometer.* The work done is measured under actual conditions of use. In the torsion dynamometer the torque is determined from the twist of a shaft or helical spring through which the power is transmitted.

65. Perpetual Motion. In its technical meaning, a " perpetual-motion machine " is not a machine which would run indefinitely if its working parts remained unimpaired, but a machine which does work without the expenditure of energy.

Thus a windmill operated by the trade-winds, or a turbine operated by Niagara Falls, or an engine working automatically by the rise and fall of mercury in a thermometer or barometer tube would not be a perpetual-motion machine, since energy would be expended in its operation. Nor is the revolution of a planet about the sun perpetual motion, since no work is done in the revolution.

It is a logical consequence of the Principle of the Conservation of Energy that " perpetual motion " as above defined is an impossibility.

66. Sources of Energy. TAIT.

a. Potential.
  1. Fuel.
  2. Food of Animals.
  3. Ordinary Water Power. (Head.)
  4. Tidal Water Power.

b. Kinetic.
  5. Winds and Ocean Currents.
  6. Hot Springs and Volcanoes.

IMMEDIATE SOURCES.
  1. Primordial Energy of Chemical Affinity.
  2. Solar Radiation.
  3. Energy of Earth's Rotation.
  4. Internal Heat of Earth.

67. The Sun as a Source of Energy. Substantially all the energy which is utilized by man is derived from the sun, e.g., that of water power, wind power, fuel, food, electrical power, though not that of tidal water power.

The heat entering the earth's atmosphere from the sun according to the determination of Abbot (1908) is 2.1 calories (gram-degrees) per square centimeter per minute. This is known as the *solar con-*

stant. Hence, energy is received at the rate of 0.15×10^7 ergs per second = 0.15 watt. This corresponds to radiation from each square centimeter of the sun's surface at a rate of about 9 H.P. The temperature of the sun's surface is probably in the neighborhood of 6000° to 7000° C.

Helmholtz (1854) explained the origin of this energy as due to slow contraction of the gaseous sun. A yearly diametral contraction of not over 250 ft. would suffice to produce the observed amount. Assuming the sun to have contracted from a sphere filling the orbit of Neptune the calculated possible radiation would not suffice for over 18,000,000 years of emission or thereabouts. On these data Kelvin has based an estimate of the maximum possible "age of the earth."

68. Historical. The truth of the doctrine of the Conservation of Energy in its complete form came to be recognized very gradually, principally during the first half of the nineteenth century. No one person can be named as its originator. Newton was apparently cognizant of the principle so far as concerns machines, and recognized the fact that there could be no gain of power by such means. The great French masters of applied mathematics in the latter part of the eighteenth century, D'Alembert, Lagrange, Laplace, and others, assumed its truth so far as concerns mechanical forces, and based their most important generalizations upon it.

Early in the nineteenth century Dr. Thomas Young introduced the term "energy" to denote mechanical energy or more strictly *vis-viva.*

In 1798 Benjamin Thompson, Count Rumford, from a study of the enormous amount of heat developed in boring a cannon showed conclusively that heat could not be an "imponderable fluid," as was then believed, but, on the contrary, is a "mode of motion," and produced by friction. It was not until a much later date, however, that this view came to be the accepted one.

Rumford's views were confirmed and adopted by Sir Humphry Davy (1799) and Dr. Thomas Young (1807), but beyond these they met with slight acceptance. Even Fourier (1822) and Carnot (1824) accepted the "caloric" theory of the material nature of heat.

Séguin, in France (1839); Mayer, in Germany (1842); Colding, in Denmark (1843), and Helmholtz, in Germany, at about the same time, all adopted the view that heat is a form of what we now call energy, and endeavored to ascertain the laws of its generation and transformations. Mayer showed a way of computing the mechanical equivalent of heat from already known data, which, though subject to criticism from a logical point of view, nevertheless proved capable of furnishing approximate results.

Joule, in England, whose work began as early as 1840, made a series of contributions to this branch of science which are of inestimable value. In 1843 he showed that the heat produced in a given circuit by a given quantity of electricity generated by a magneto-electric machine was the same in amount as that which would be produced by direct transformation into heat of the work consumed in producing the current of electricity by the machine. He also obtained from his measurements a value for the mechanical equivalent. Shortly thereafter he determined this constant by direct measurement under different conditions of the work done by friction and of the resulting heat. Using various methods of producing heat by the expenditure of mechanical work, all of which were found to give a like value of the mechanical equivalent, he showed this to be independent of the particular manner in which the work was done. In 1850 he published a paper of great completeness giving a value of $J=772$ ft.-lb. which continued to be accepted for almost thirty years.

In 1847, Joule, in Manchester, and Helmholtz, in Berlin, each read a paper now regarded as epoch-making in importance, in which the principle of the conservation of energy in its full breadth was clearly set forth. But their views were not accepted in either country, and Poggendorff's *Annalen*, the leading physical journal in Germany, declined to publish the paper of Helmholtz. But largely through the ability and influence of William Thomson (afterwards Lord Kelvin) and Du Bois Reymond, the new views rapidly gained acceptance.

The reverse transformation of heat into mechanical work was investigated by Hirn, Rankine, Kelvin, Clausius, and others, showing the same quantitative relationship to exist between heat transformed and the resulting work.

The expression "Conservation of Energy," replacing the earlier and confusing expression "Conservation of Force," was introduced by Rankine.

The laws of relationship of electrical quantity and chemical decomposition were discovered by Faraday (1833), those relating to the heating effects of electricity by Joule (1841). More recently the laws relating to heat produced by chemical action have been determined by Favre and Berthelot.

All the results obtained, without exception, have been such as to confirm and extend the conclusion that energy is neither created nor destroyed in any of its varied transformations.

69. Dissipation of Energy. *All energy tends to pass from a higher form to a lower one, and ultimately to assume the form of uniformly diffused heat.*—KELVIN.

Hence, although the total energy of a system remains constant, the amount of its *available* or *free* energy continually diminishes toward zero. Thus, it is easy to transform completely the total amount of mechanical energy of a system into heat energy, but it is impossible to transform the total amount of heat energy of a system into mechanical energy. The total electrical energy of a system can be transformed into heat, but the total heat energy of a system cannot be transformed into electrical energy.

For example, it can be shown that with a perfect steam or other heat engine, if the vapor enters the cylinder at a temperature of 200° C., and leaves it at a temperature of 110° C., the efficiency can be only 19 per cent. With the actual engine it is much lower.

70. Application of Principle to Cosmical Phenomena.—The principle of the dissipation of energy applies to the physical universe as far as we know it, whence we conclude that the available energy of the universe tends toward zero. There is, however, no evidence of retardation of planets or comets such as would be caused by a resisting medium in space.

71. Transmission of Energy. 1. Transference of body possessing energy. 2. Waves.

72. Waves. In any wave the wave-form alone progresses, while the particles at any point merely assume an oscillatory or rotary motion.

Waves are of three kinds: *Longitudinal, transverse,* or *torsional,* according as the motion of the particles is parallel to the line of transmission of the wave, transverse to it, or twisting about it as an axis.

The following classes of wave may be produced in ordinary matter: (*a*) Gravitational waves in liquids, due to displacement of level. (*b*) Capillary waves or surface ripples, due to surface tension of the liquid. (*c*) Pressural waves in body of solid, liquid or gas, due to volume elasticity. (*d*) Distortional waves in solid, due to elasticity of shearing, bending, twisting. Of these (*a*) and (*b*) are necessarily transverse waves (*c*) longitudinal, (*d*) either transverse or torsional.

Water waves are transverse; the sound wave is longitudinal.

In the ether, electromagnetic waves exist, of which light waves and the electric waves, utilized in space telegraphy, are examples. These are due to electrical and magnetic properties of the ether which are analogous to elasticity and mass in ordinary matter. Electromagnetic waves are transverse in their character.

Curvilinear Motion. Rotation

73. General Principles. Definitions. Curvilinear motion is produced by the continuous action of a deflecting force upon a moving particle or mass. The deflecting force may vary in any manner whatever, both as to magnitude and direction.

An important case is that in which the deflection is always toward the same point, in which case the deflecting force is called a *centripetal force.* Such a point is called a *center of force.* The so-called " tangential or projectile force " in reality is not a force, but the momentum of the particle.

It will readily be seen that the nature of the orbit traversed by the particle will depend upon the manner in which the magnitude and direction of the deflecting force vary. Hence, from the form of the orbit the law of variation of the deflecting force can be determined.

Newton showed that if the orbit of a body is a conic section the line of action of the deflecting force must always pass through a focus of the curve, and, furthermore, must vary in magnitude from point to point of the curve in the inverse ratio of the square of the distance of such point from that focus; and conversely. This is the case with all members of the solar system. All the planets move in ellipses of small eccentricity with the sun situated in one focus of the ellipse. Many comets also move in orbits which are elliptical, but of great eccentricity. Most comets move in sensibly parabolic orbits, and some, apparently, in hyperbolic orbits.

Since the centripetal force acts to draw the revolving body inward toward the center of force, it follows from the third Law of Motion that this center sustains a radially outward pull equal in magnitude to the centripetal force. It is this tendency of the center of motion to move radially outward which led to the conception of the definite existence of a *centrifugal force.* In fact, there is no such force. It is only the reaction on the axis of the centripetal force, which pulls the body and the center equally towards one another, though the fixation of the center prevents this from moving. The revolving body always tends to move tangentially in virtue of its inertia.

As the tangential momentum, however, tends to increase the radius of revolution, the term " centrifugal force " has come to be habitually used by engineers and others as denoting a component of the momentum which is opposite to the centripetal force, and, necessarily, equal to it.

No comet has actually been proved to move in any curve other than an ellipse. Parabolic orbits are usually assumed for comets of very long period on account of the greater simplicity of computation.

74. Centripetal Force in Circular Orbit. The case most frequently occurring in the ordinary applications of Physics is that of a particle or body revolving in a circle about a center of force situated at the center of the circle. The curvature of the circle being constant, the centripetal force in such an orbit is of constant intensity. Denote it by F.

To determine its value let C (Fig. 8) be the center of the circle in whose circumference the particle revolves, PR the space over which the particle would pass in time t in virtue of its tangential velocity, PM the space which the particle would traverse in the same time in virtue of the action of the centripetal force alone. Let PQ be the chord of the arc in which the particle moves. From the geometry of the circle, $PQ^2 = PM \times PD$.

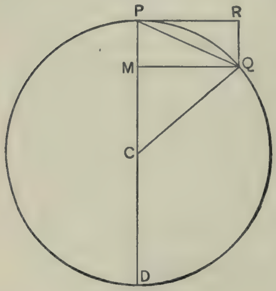

FIG. 8.

Denote by f the centripetal acceleration, and by v the constant orbital velocity of the particle. If s is the space which would be traversed toward C in time t under the influence of the centripetal force, $\frac{d^2s}{dt^2} = f$. But $s = PM = \frac{PQ^2}{PD}$. Also for indefinitely small arcs, the chord and arc coincide, whence $PM = s = \frac{v^2t^2}{2R}$ and $\frac{d^2s}{dt^2} = \frac{v^2}{R} = f$. Hence

$$F = Mf = M\frac{v^2}{R}.$$

Hence, *the centripetal force varies directly as the mass of the particle and the square of the velocity, and inversely as the radius of the orbit.*

Evidently M may be written in either gravitation or absolute units, in which case F will be expressed in similar units.

The centripetal force of any extended body or system of bodies is the same as if the total mass were concentrated at its center of mass. Hence the preceding demonstration can be applied immediately to such cases.

The formula may also be put into the following form:

Let T be the time of revolution; then

$$vT = 2\pi R,$$

and

$$v=\frac{2\pi R}{T}.$$

Hence

$$F=M\frac{v^2}{R}=M\frac{4\pi^2 R}{T^2}.$$

Hence, for bodies revolving in circular orbits, *the centripetal force varies directly as the mass and the radius of the orbit, and inversely as the square of the time of revolution.*

74a. Centrifugal Force in Circular Orbit. Alternative Demonstration. Let PQ be the elementary arc which the particle describes in time dt. In virtue of its tangential velocity alone in time dt it would traverse the space $PB=vdt$. BQ is, therefore, the space ds through which in time dt the particle is drawn toward the center C under the action of the constant centripetal force F. Calling f the centripetal acceleration and remembering that $PB^2=BQ\times BE$, we have

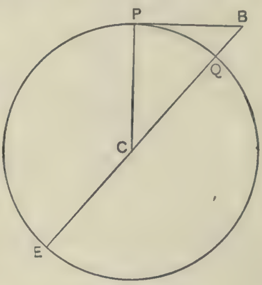

FIG. 8*a*.

$$v^2dt^2=ds(2R+ds)$$

and

$$ds=\frac{v^2dt^2}{2R+ds}=\frac{v^2dt^2}{2R}$$

in the limit. But $ds=\frac{1}{2}fdt^2$, since the centripetal force is constant, whence

$$f=\frac{v^2}{R}.$$

75. Velocity in Orbit. The uniform velocity with which a body revolves in a circular orbit is equal to that which the centripetal force would generate by its constant action upon the body through half the radius of the orbit. For

$$f=\frac{v^2}{R};$$

whence

$$v=\sqrt{fR}$$

Suppose the centripetal force to impel the body from a state of rest until it attains the velocity v. Call s the space described. Then

$$v=\sqrt{2fs};$$

whence

$$\sqrt{fR}=\sqrt{2fs},$$

and

$$s=\frac{R}{2}.$$

76. Body Revolving in Vertical Orbit. That a body may revolve in a vertical circular orbit, its centrifugal force must, at least, be equal to its weight. To find the minimum velocity requisite for this, put $W=F$ in the general equation. Then

$$W=\frac{Wv^2}{gR}, \quad \text{and} \quad v=\sqrt{gR}.$$

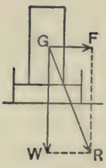

FIG. 9.

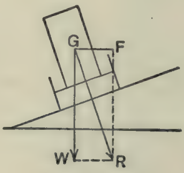

FIG. 10.

Illustration. Centrifugal Railway.

77. Practical Illustrations. Centrifugal dryer, blower, pump, cream-separator; centrifuge; bursting of fly-wheels; vehicle on curved road (Figs. 9 and 10); depression of inner rail on curve.

78. Conical Pendulum.—In the conical pendulum the ball P must assume such a position that the cord AP is in the line of action of R, the resultant of the weight and the centrifugal force of the ball.

Under these circumstances $\tan\theta=\frac{F}{W}=\frac{r}{h}$, denoting by r the radius of the circle of revolution of P and by h the altitude of the cone described by the suspending cord. The time of rotation is $2\pi\sqrt{\frac{h}{g}}$. For the centrifugal force $F=W\frac{r}{h}=\frac{W}{g}\,\frac{4\pi^2 r}{T^2}$, and $T=2\pi\sqrt{\frac{h}{g}}$.

Applications.—Watt's governor for steam engine; regulating mechanism for chronographs; driving clock for equatorial telescope.

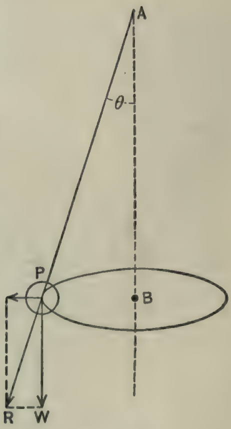

FIG. 11.

79. Equilibrium Surface of Liquid Rotating about Vertical Axis. This is a paraboloid of revolution. At every point on the free liquid surface that surface must be at right angles to the resultant of the weight W of the particle and its centrifugal force F. Hence (Fig. 12)

$$\tan\theta = \frac{W}{F} = W \div \frac{W4\pi^2 r}{gT^2} = \frac{gT^2}{4\pi^2 r};$$

and

$$r\tan\theta = \frac{gT^2}{4\pi^2},$$

which value is constant for all points on the liquid surface. Hence

$$r\tan\theta = Ac,$$

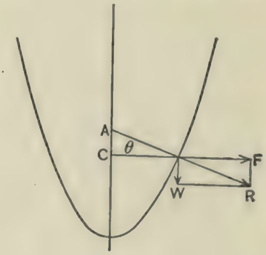

FIG. 12.

the subnormal of the curve, is a constant, which is a well-known property of the parabola.

80. Centrifugal Force at Equator. The magnitude of the centrifugal force at the equator due to the earth's rotation is found by substituting the proper values of R and T in the general equation.

The equatorial radius of the earth = 6378250 m. (Clarke, 1880), and the value of g at the equator as measured is 9.78 m. Hence

$$F = M\frac{v^2}{r} = W\frac{4\pi^2 R}{gT^2} = W\frac{4 \times (3.1416)^2 \times 6378250}{9.7807 \times (86164)^2} = W\frac{1}{288.4}.$$

Hence the effect of the centrifugal force at the equator is to diminish the weight of every body by its $\frac{1}{288}$ part. Since F varies as v^2, a velocity of approximately seventeen times the present would generate a centrifugal force at the equator entirely counterbalancing the action of gravity.

81. Variation of Centrifugal Force with Latitude.—It will appear from a consideration of Fig. 13, that the value of the centrifugal force in latitude λ, represented by PF, will be to its value at the equator in the ratio of the respective radii of revolution of a point in these latitudes. Hence, $F = F_0 \cos\lambda$.

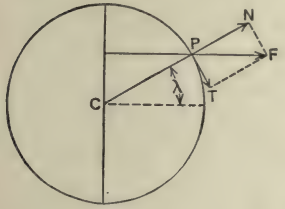

FIG. 13.

The diminution of weight due to centrifugal force in latitude λ is represented by the normal component PN in the diagram. Calling this N, we have $N = F_\lambda \cos\lambda = F_0 \cos^2\lambda$.

It will also be seen that the existence of a tangential component, PT, of the centrifugal force acts to produce a deviation of the plumb-line toward the equator For latitude 45° the value of this can be shown to be 11′ 30″.

Other important phenomena in which the centrifugal force due of the earth's rotation is concerned are those of cylcones and storms in general, and the tides. A suggested deviation of river courses is doubtful.

82. Effect of Centrifugal Force on Figure of the Earth. Spheroid of Equilibrium. The tangential component PT acts to produce motion in a movable body, as, for example, the water of the ocean, from the poles toward the equator. Hence with a fluid mass, such as the earth must have been at a former remote period, there would result a heaping up of matter in the equatorial regions, so that the earth thus rotating could no longer be spherical.

The actual form assumed by such a fluid rotating mass must be such that the surface at every point is normal to the resultant of the weight of the particle and the centrifugal force. If P, Fig. 14, is any such point PR, the resultant of the weight PW and the centrifugal force PF must be normal to the surface at P.

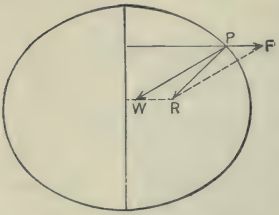

FIG. 14.

Maclaurin (1740) showed that an oblate spheriod fulfills this condition. It has since been shown by Jacobi and Poincaré that there are several other possible figures of equilibrium.

From the mass of the earth and its period of rotation it is possible to calculate the amount of the polar flattening.

The ellipticity of the earth as measured is $\frac{1}{300}$. That of Jupiter, whose period of revolution is only 9 h. 55 m., is $\frac{1}{17}$.

83. Plateau's Experiment. Sphere of oil immersed in mixture of alcohol and water of same density, becomes spheroidal when rotated, and finally throws off equatorial rings if the velocity is increased.

84. Theories of Evolution of Solar System. (*a*) Nebular Hypothesis of Kant (1755) and Laplace (1796). Difficulties in the way of its acceptance. (*b*) Planetesimal or Spiral-nebula Hypothesis of Chamberlain (1905).

85. Kelvin's Estimate of "Age of Earth." Ellipticity (assumed to be substantially unchanged since solidification) in connection with increase in length of day from tidal retardation leads to the conclusion that some 20,000,000 years have elapsed since solid crust was formed. Estimates based on an entirely different class of data lead to a value of the same order of magnitude, as has already been stated in connection with the energy of solar radiation.

Objections. Plasticity of earth has probably allowed a subsequent change in ellipticity Amount of retardation is not certain.

The action of tidal friction to lengthen the day was suggested by Delaunay in 1866 as an explanation of the results obtained by Adams (1859) relative to the secular acceleration of the mean motion of the oon. From these it was shown by Adams that the effect is such that the amount by which the earth regarded as a timekeeper would in a century fall behind a perfect clock rated at the beginning of the century is twenty-two seconds. The observational data are, however, somewhat uncertain. The astronomical data as computed by Newcomb would reduce the above result to 8.3 seconds.

The results of recent studies of radioactive phenomena, however, lead to the conclusion that unquestionably a far greater age must be assigned on the earth than that deduced by Kelvin.

86. Rotation of Rigid Bodies. The effect of an unbalanced centrifugal force is two-fold, tending (1) to shift the axis of rotation as a whole, i.e., to produce a translatory motion of the axis, and (2) to produce angular deviation of the body. (See Figs. 15, 16.)

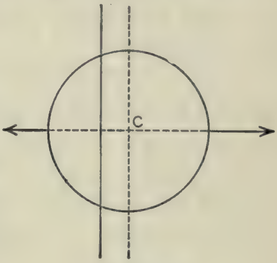

FIG. 15.

87. Free Axes. Principal Axes. It will be seen from the figures that, if the axis of revolution passes through the center of mass, the tendency to translatory motion disappears. That there may also be no tendency to angular motion, the axis of rotation must coincide with an axis of symmetry of the body. (See Fig. 17). If the axis of rotation is parallel to an axis of symmetry, but does not pass through the center of mass, there will evidently be no tendency to angular deviation. An axis about which a body may revolve without causing any tendency to angular deviation is called a *principal axis.* Any axis about which a body may revolve without producing a tendency to either angular deviation or translation of the axis of

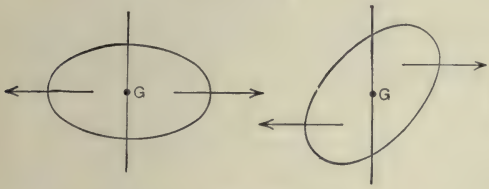

FIG. 16.

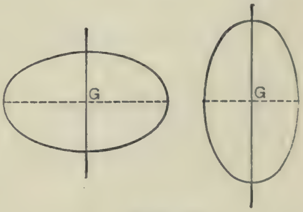

FIG. 17.

rotation is called a *free axis.* The free axes are evidently principal axes passing through the center of mass.

In an ellipsoid with three unequal axes, these are the free axes. In a right elliptical cylinder the free axes are the axis of the cylinder and the major and minor axes of the elliptical middle section of the cylinder. Any diameter of a sphere is a free axis; also, any diameter of the equator of an oblate or prolate spheroid, together with its polar axis.

Practical Applications in Machinery. Balancing of fly-wheels and other rotating pieces; balancing of crank on driving-wheel of locomotive.

88. Axis of Stable Rotation. An inspection of Fig. 16 will also show that a body is in stable equilibrium only when rotating about its shortest diameter.

In machinery several practical considerations frequently require that a rotating piece shall be rotated about a longer rather than a shorter axis of symmetry. This, however, is not a case of free rotation, and the rigidity of the shafting is made such as to counterbalance any centrifugal couple that may be generated.

89. Analogies between Translatory and Rotary Motion. Some further characteristics of the rotary motion and energy of rigid rotating bodies are most readily considered in this connection.

It is easily seen that the effect of a constant unbalanced torque is to produce a uniformly accelerated rotation in a mass, and, hence, that relations hold between α, ω, and θ, identical with those which obtain in the case of uniformly accelerated translatory motion.

Denote the angular velocity of such a body by ω, its angular acceleration by α, and the angle described in time t under the action of the constant torque by θ. Then

$$\frac{d\theta}{dt}=\omega, \quad \frac{d\omega}{dt}=\frac{d^2\theta}{dt^2}=\alpha.$$

From this it follows that

$$\omega=\alpha t, \qquad \theta=\tfrac{1}{2}\alpha t^2, \quad \omega=\sqrt{2\alpha\theta},$$

which formulæ are of the same character as those already proved for uniformly accelerated translatory motion; viz.:

$$v=at, \qquad s=\tfrac{1}{2}at^2, \quad v=\sqrt{2as}.$$

It will, furthermore, be seen that a simple relation exists between the work done upon a rotating mass by a torque T and the resulting kinetic energy; i. e., $T\theta=\frac{1}{2}\omega^2 I$. For calling F, r, respectively, the force and arm of the tongue, we have $T\theta=Fr\theta=FS=\frac{1}{2}\omega^2 I$. (See § 46.)

It will be noted that the moment of inertia, I, takes the same place in the dynamics of rotation that the mass M takes in the dyramics of translation. This will be apparent from the comparisons in the following table:

Translation	*Rotation*
$v=at$	$\omega=\alpha t$
$s=\frac{1}{2}at^2$	$\theta=\frac{1}{2}\alpha t^2$
$v=\sqrt{2as}$	$\omega=\sqrt{2\alpha\theta}$
$F=Ma$	$T=I\alpha$
$Ft=MV$	$Tt=I\omega$
$K=\frac{1}{2}MV^2$	$K=\frac{1}{2}I\omega^2$
$FS=\frac{1}{2}MV^2$	$T\theta=\frac{1}{2}I\omega^2$

90. Ballistic Pendulum. A heavy block of wood is suspended on knife-edges. A bullet whose velocity is to be measured is fired into it, in a line at right angles to the axis of suspension, producing an angular velocity ω. The deflection of the pendulum is measured and ω determined from this. Calling m the mass of the bullet, v its velocity, I the moment of inertia of the pendulum, k the distance from the axis of suspension to the line of entrance of the bullet, $I\omega=mvk$ and $v=\frac{I\omega}{mk}$. In strictness the mass of the bullet should be taken into account in the value of I, but this is so small that practically it is unimportant.

In order not to jar the support of the pendulum the line of fire should pass through its *center of percussion* as will be explained later.

The moment of inertia may either be calculated or determined experimentally.

91. Moment of Momentum. Angular Momentum. The moment of a force F relatively to any point O is Fr_0 where r_0 is the normal drawn from O to the line of action of F. In like manner if a mass moves with a velocity v, its moment of momentum relatively to O is $T_0=mvr_0$, r_0 being normal to the direction of v. This quantity, also called angular momentum, represents the rotary effect produced by the momentum mv.

92. Conservation of Angular Momentum. The angular momentum of a system of bodies is not in any way altered by the mutual actions of the masses composing that system.

Let m_1, m_2, Fig. 18, be two masses, and let their angular momenta be taken relatively to an axis passing through any point, as O, normal to the plane of the paper. The action of m_1, m_2 on each other will be along the line joining them. As action and reaction are equal and opposite the force F_1 with which m_1 is drawn toward m_2 will equal the force F_2 by which m_2 is drawn toward m_1. Under the action of this mutual attraction there will be generated in each in time t a quantity of momentum such that $Ft=m_1v_1=m_2v_2$. But as will be seen from the figure $m_1v_1r_0$ is the angular momentum of m_1 relatively to O and $m_2v_2r_0$ that of m_2 relatively to the same point. Since these are in opposite directions the resultant angular momentum $=0$. Hence they can in no way alter the previously existing angular momentum of the system.

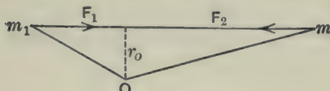

FIG. 18.

What is true of two masses m_1m_2 is equally true of any number, for the effect of any one mass upon any other in the system can be dealt with in the same manner, and since the effect of every individual mass upon every other is to produce equal and opposite angular momenta with, therefore, a resultant of zero, the total mutual effect of the masses composing the system must also be zero.

93. Conservation of Areas. It follows from the preceding demonstration that if we suppose a rotating mass m_1 with radius r_1 and velocity v_1 so to move that its radius changes to r_2 the velocity will become v_2 having a value such that $m_1v_1r_1=m_1v_2r_2$. Hence, considering a very brief time dt, we have $r_1v_1dt=r_2v_2dt$. But the first member of this equation is double the area of the elementary triangle described in time dt by the mass when its radius is r_1 and the second double the area described by it when its radius has become r_2. Hence the areas traversed by the radii vectores in equal times are the same.

Because of this fact the conservation of angular momentum has sometimes been called the "conservation of areas." Kepler's Second Law is an illustration of it.

It will be seen from what has been said that if a portion of a revolving mass is transferred from a point near the axis of revolution to one farther removed from it the angular momentum of the mass is diminished by the same amount as that by which the angular momentum of the portion moved is increased. And while the movable part is gaining speed it reacts on the mass as a whole.

94. Composition of Rotations. If the angular velocities of two rotations are represented by the two adjacent sides of a parallelogram, the diagonal of that parallelogram will represent the angular velocity of the resultant.

Let AB, AC (Fig. 19) represent component angular velocities ω_1, ω_2. Consider any point P whose coordinates referred to AC, AB are x, y. Draw PM, PN normal to AB, AC. The downward linear velocity of P due to the rotation AB is $\omega_1 PM = \omega_1 x \sin BAC$. The upward linear velocity of P due to the rotation AC is $\omega_2 PN = \omega_2 y \sin BAC$. For points on the resultant axis these opposite rotations must be equal. Placing $\omega_1 x \sin BAC = \omega_2 y \sin BAC$, we have $\frac{x}{y} = \frac{\omega_2}{\omega_1} = \frac{AC}{AB}$, which is the equation of the diagonal AD. This diagonal, therefore, represents the resultant axis in direction.

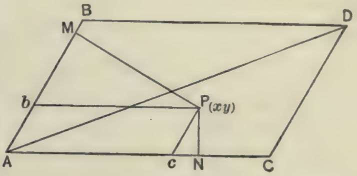

FIG. 19.

AD furthermore represents the resultant rotation in magnitude. For, consider the motion of the point C. Its linear velocity due to the resultant rotation must be equal to that due to the simultaneous action of the two components. But the velocity due to rotation $AC = 0$. That due to $AB = AC \sin BAC \times \omega_1$. That due to $AD = AC \sin DAC \times \omega_r$, calling ω_r the resultant angular velocity. Hence, as these last must be equal,

$$AC \sin BAC \times \omega_1 = AC \sin DAC \times \omega_r \text{ or}$$

$$\omega_r = \omega_1 \frac{\sin BAC}{\sin DAC} = AB \frac{\sin BAC}{\sin DAC} = AD.$$

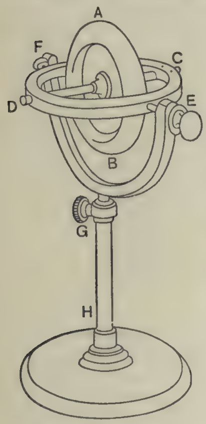

FIG. 20.

95. Gyroscope or Gyrostat. Referring to Fig. 20, let DC, the axis of spin of the disk, be designated by X, the axis FE by Y, the vertical axis HG by Z. Assume that there is perfect freedom of motion about each of these axes. With the instrument balanced so that its center of mass lies at the center of the rotating disk AB, the axis of spin remains parallel to itself in whatever manner the instrument as a whole is moved. If, however, the instrument is unbalanced, as e.g., by hanging a weight from D, a precessional rotation takes place about Z, instead of rotation about Y such as would occur under like circumstances were the disk at rest.

The precessional motion results from the combination of two rotations, the spin of the disk about X and the rotation about Y due to the couple produced by the suspended weight. Assuming the direction of spin to be right handed as seen from *D* we may represent the angular velocity about *DC* by *Ox*, Fig. 21.

The angular velocity due to the couple produced by the suspended weight may be represented by *Oy*. Then *Or* will represent the resultant rotation, in magnitude and direction. The axis of spin will, therefore, move in such manner as to tend to approach *Or*, as indicated by the arrow in *XOY*, remaining, however, in a horizontal plane, rotating about *OZ*, Fig. 21, or *HG*, Fig. 20. But as this rotation carries the axis *FE* with it, the direction of *Oy* and, hence, of *Or* is constantly changing, performing a rotation about *OZ*; whence the precessional movement is continuous.

It is easily seen from the construction of Fig. 21 that if either the direction of spin about *DC* or that of the gravitational couple about *FE* is reversed the direction of precession will likewise be reversed.

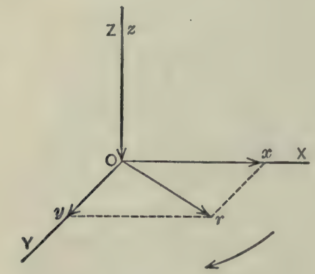

FIG. 21.

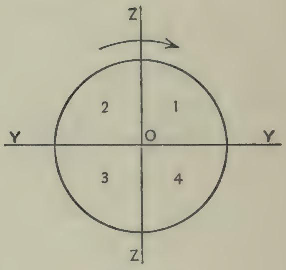

FIG. 22.

The following is a general explanation on dynamical principles of the phenomena just considered. Fig. 22 represents a disk rotating right-handedly about an axis through *O* perpendicular to the plane of the paper as an axis of spin. Suppose that a second rotation is given to the disk, about *YOY*, the upper half moving toward the eye. The particles in quadrant 2 on account of the spin about *O* are moving upwards and increasing their distance from *YOY*. Hence on account of their momentum their effect is to produce a tendency in 2 to move away from the observer. In quadrant 1 on the other hand, the particles are descending towards *YOY*, and on account of their momentum the diminution in their radii of revolution about *YOY* will cause a pressure throughout 1 towards the eye. In like manner in quadrant 4 there will be a pressure toward the eye and in quadrant 3 away from it. Hence a rotation will be produced about *ZOZ* as an axis.

The action described above is analogous to the case of a ball whirled by a string, which, when the string is lengthened, so that its radius of revolution is increased, has its velocity correspondingly slackened, and which, if it were being pushed forward, would press backward in opposition. The conception of the action may also be helped by supposing the disk to be a rotating disk of liquid contained in a flat box, mounted like the gryoscope.

The several actions take place in accordance with the principle of the conservation of angular momentum. (See § 92.)

The great resistance opposed by a gyroscope to sudden angular deviation of its axis of rotation is due to the action just explained. The combination of couples produces a resultant couple such as would generate a precessional motion, and, if the impressed force is such as to prevent this, the resisting effect due to the momentum of the wheel becomes very great.

Other gyroscopic phenomena are to be explained according to the principles just laid down.

Thus, with diminishing speed the precession is faster because Ox becomes less in proportion to Oy. If the weight on D is increased or if this point is pushed down the rate of precession is increased; if pushed up, diminished. If the axis of spin is tilted the precession is faster as the effective value of Ox is less. If the precession is accelerated by external force, the center of mass of the system rises; if the precession is retarded, the center of mass descends. The explanation of this may be seen by combining a rotation about the axis Oz with Or. If the axis HG is fixed, so as absolutely to prevent precession, the added weight will produce rotation about FE, as if the disk were not rotating. There will be a torsional stress, however, in the vertical axis due to the precessional tendency. Retardation of precessional motion by friction of the axis HG will cause a gradual depression of the axis of spin.

An excellent rule to remember is that with a gyroscope the axes tend to become parallel and to assume the same direction.

96. Phenomena and Applications of Gyroscope. Rolling hoop; bicycle; game of diabolo; rifled guns; steadying gyroscope on vessel to prevent rolling; steadying gryoscope and steering apparatus in Howell torpedo; Obry's steering apparatus in Whitehead torpedo; Brennan's mono-rail system. Also various devices proposed for determining latitude at sea; for transferring directional line, as, e.g., the meridian, from surface to bottom of mine, etc., but none of them as yet practical.

The gyroscopic compass has recently come into somewhat extended use, particularly on war vessels. The compass of Sperry is preferred in the United States Navy, that of Anschütz abroad.

97. Astronomical Gyroscopic Phenomena. Parallelism of axis of earth to itself, causing phenomena of seasons. Precession of the equinoxes caused by action of sun on equatorial protuberance of earth, with period of about 26,000 years. Nutation, caused by action of moon on equatorial protuberance, with period of about nineteen years. Periodic variations of latitude investigated by Chandler, cause uncertain.

Application of principle of gyrostat by Kelvin to explain hypothesis of elastic-solid luminiferous ether. Kelvin's hypothetical gyrostatic elastic atom.

98. Top. The various precessional phenomena of the top are to be explained upon the same principles. The rise of a top from an inclined to a vertical position is due to the combination of precessional motion with the rolling motion of the peg. The roll tends to carry the top along faster than the precession and so to accelerate this, thereby causing the center of mass of the top to rise.

Maxwell's dynamical Top. Newton's geometric Top. Griffin's stone-pulverizer.

Periodic Motion

99. Definitions. A particle possesses a periodic motion when it successively traverses the same path, returning to any given point after having passed through a complete cycle of changes. Such motion may take place in a closed curve, an arc, or a straight line. The time occupied in completing one cycle is called the *period.*

The earth revolves about the sun in an elliptical orbit, with a period of one year; a " seconds " clock-pendulum vibrates in a circular arc, with a period of two seconds; a body suspended by an elastic cord executes a vibrating periodic motion in a vertical line, with a period equal to the time occupied by one complete " up and down " motion.

Any periodic motion can be represented by a curve constructed with times as abscissas and the corresponding displacements of the particle as ordinates. Such a curve is evidently periodic. When thus constructed it is a *curve of spaces* or *displacements.*

A curve representing periodic motion may also be constructed in which times are abscissas and the corresponding velocities are ordinates. This is a *curve of velocities.*

The curve of velocities can always be derived from the curve of spaces, and conversely, since $v=\frac{ds}{dt}$.

100. Simple Harmonic Motion. An extremely important case of periodic vibratory motion is that known as *simple harmonic motion.*

Referring to Fig. 23, let us suppose a particle P to revolve uniformly in the circle there shown, whose radius is r, and likewise that a second particle P_1 moves to and fro along the vertical diameter of that circle in such manner as always to coincide in position with the projection of P on that diameter. The motion of the vibrating particle is a simple harmonic one.

The position in its path of the vibrating particle at any instant is denoted by what is called the *phase* of the vibration. This term is defined more precisely as follows: "The phase of a simple harmonic motion at any instant is the fraction of the whole period which has elapsed since the moving point last passed through its middle position in the positive direction." The phase is measured by the angle θ in Fig. 23.

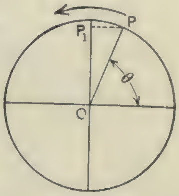

Fig. 23.

When a particle after displacement executes a vibration of this character the restoring force at each point must be proportional to the displacement.

The acceleration at any point P_1 is given by the equation $\frac{d^2y}{dt^2}=a.$ But $y=r\sin\theta=r\sin\omega t$, calling ω the angular velocity and t the time corresponding to the phase-angle θ; that is, the time occupied by the vibrating particle in passing from O to P_1. Then $\frac{dy}{dt}=\omega r\cos\omega t$ and $\frac{d^2y}{dt^2}=-\omega^2r\sin\omega t=-\omega^2y$; whence $a\propto y$, the displacement of the particle; and the same must be true of the restoring force, since $F=Ma$. The opposite sign of a and y indicates that the acceleration is opposite in direction to the displacement.

The converse of the proposition is likewise true, since this law of force can produce only one mode of variation of the acceleration.

If T be the time of one complete revolution of P and hence also the time of one complete vibration of P_1, $T=\frac{2\pi}{\omega}$ and $\omega=\frac{2\pi}{T}$.

It has just been shown that $a=\omega^2y$ (dropping the minus sign as indicative only of direction) whence $\omega^2=\frac{a}{y}=\left(\frac{2\pi}{T}\right)^2$ and

$$T=2\pi\sqrt{\frac{y}{a}}=2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}.$$

It also appears from this expression that *simple harmonic vibrations are isochronous,* since the ratio of y to a in any particular case is a constant and independent of r.

101. Equations of Harmonic Curves. It is apparent that the movement of a particle possessing a simple harmonic motion is represented by the equation $y=r \sin \theta$, or $y=r \sin \omega t$, where $\omega=\frac{2\pi}{T}$, y being the displacement. Hence, denoting times by abscissas, the equation of the curve of displacement is $y=r \sin \omega x$, which is the equation of a sinusoid, r being the amplitude of vibration.

The corresponding curve of velocities is readily found. Since $v=\frac{ds}{at}$ and in this case $s=y=r \sin \omega t$, $v=\omega r \cos \omega t$. Hence, taking velocities as ordinates and times as abscissas, the equation of this curve is $y=\omega r \cos wx$. This is also a sinusoid, but displaced along the axis of X relatively to the former curve by a distance $\frac{\pi}{2}$ corresponding to a quarter of a cycle or vibration period.

Apart from the demonstration it is easily seen that the maximum velocity of the vibrating particle corresponds to displacement $=0$, while the maximum displacement corresponds to velocity $=0$.

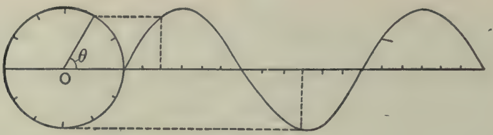

FIG. 24.

The equation $y=r \sin \omega t$ assumes that for $y=0$, $t=0$, i.e., the particle is assumed to be at the middle of its path at the beginning of the time interval considered. Hence, the curve representing the motion passes through the origin. If when $t=0$ the particle occupies any other position, the phase being, for example, an angle δ in advance of the zero position, the equation may be written $y=r \sin (\omega_t+\delta)$ or $y=r \sin (\omega x+\delta)$, which is a more general form than the preceding. δ is called the *angle of epoch.*

Fig. 24 illustrates the construction of a curve representing a simple harmonic vibration. The abscissas are made proportional to times, and hence to arcs on the reference circle, i.e., to the angle θ. The ordinates are made proportional to displacements parallel to Y, i.e., to $\sin \theta$.

In the study of vibrations it is often more convenient to measure the phase-angle from the axis of Y rather than from X, and to use cosine rather than sine functions. In this case a vibration parallel to Y is represented by the equation $y=r\cos(\omega t+\delta)$ and the corresponding harmonic curve by the equation $y=r\cos(\omega x+\delta)$.

102. Permanent Record of Vibrations. Trace by pencil carried at end of elastic vibrating bar against sheet of paper, moved uniformly at right angles to direction of vibration. Curves drawn on smoked glass by style attached to uniformly moved vibrating tuning fork, or on smoked glass carried by vibrating tuning-fork, the style being moved uniformly in a straight line.

102a. Wave Length; Amplitude. The *length* of a wave is the distance between two points at which the phase of vibration at the same instant is the same. For example, the length of a water-wave is the distance between the uppermost points of two successive crests or between the lowermost points of two successive troughs. Likewise, the length of a sound wave is the distance from one point of maximum condensation or rarefaction to the following one.

From the mode of generation of a wave it will be seen that the wave-length is the distance traveled by the wave during the time of one complete vibration of the body originating the wave and of the particles composing the wave itself. Hence, calling v the velocity of propagation of the wave, λ the wave-length and n the corresponding frequency of vibration, $\mathbf{v}=\mathbf{n}\lambda$, an equation which is true for all kinds of waves.

The *amplitude* of a wave is the maximum displacement of the particles composing it and is, therefore, half the extreme range of vibration of a particle.

The height of a transverse wave, as, for example, a water-wave, is double the amplitude.

103. Equation of Wave Motion in Elastic Medium. From the equation of the harmonic curve we can derive the equation of a simple wave propagated in an elastic medium, as, for example, the sound-wave.

From the principles of wave motion it follows that the instantaneous value of y at time t for any point of the wave whose abscissa is x is the same as the value which it had at $x=0$ (the source of the wave disturbance) at a previous time earlier than that under consideration of the interval which has elapsed between the starting of the wave from $x=0$ and its arrival at x.

Denoting by T, as before, the time of one complete vibration and by m, the number of cycles (in general not an integer) that have elapsed since the wave started from the origin, this interval $=mT$. Hence

$$y=r\sin\omega(t-mT)=r\sin\frac{2\pi}{T}(t-mT)=r\sin 2\pi\left(\frac{t}{T}-m\right).$$

If we denote by λ the length of the wave under consideration, $m\lambda=x$ and $m=\frac{x}{\lambda}$, since m waves have been generated while the disturbance was traveling from $x=0$ to x. Hence $y=r\sin 2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)$, which is the form in which the equation of the elastic wave is usually given.

As the velocity of propagation of the wave is $v=n\lambda=\frac{\lambda}{T}$, ($n$ = frequency) this equation may also be written in the form

$$y=r\sin 2\pi\left(\frac{t}{T}-\frac{x}{vT}\right).$$

It will be seen on consideration that the equation of the wave is that of the curve shown in Fig. 24. The axis of Y at the instant under consideration, t, for which the phase is $\theta=\frac{2\pi t}{T}$, must pass through the point for which the ordinate is $y=r\sin 2\pi\frac{t}{T}$. This defines the location on X of $x=0$ at that instant.

104. Combination of Parallel Vibrations and of Harmonic Curves. It is evident that for each value of x on the resultant curve the ordinate will be equal to the algebraic sum of the ordinates of the component curves. Hence if the equations of these are $y_1=r_1\sin(\omega_1 x+\delta_1)$ and $y_1=r_2\sin(\omega_1 x+\delta_2)$ the equation of the resultant curve will be $y=r_1\sin(\omega_1 x+\delta_1)+r_2\sin(\omega_2 x+\delta_2)$.

This equation represents a periodic curve, since if $\omega_1 x$, $\omega_2 x$ both be increased by 2π or $n2\pi$, the values of y recur. The period of the resultant curve depends upon the ratio of ω_1 to ω_2. If this reduced to its lowest term is $p:q$ the resultant period will be the time taken to make p vibrations of the first or q vibrations of the second component.

In general the nature of the results is better apprehended from a study of the graphical combination of such curves than from analysis.

Various methods have been described for drawing such compound harmonic curves mechanically.

105. Applications. Curves drawn with tuning-forks, the smoked plate carried by one fork, the style of another. Beat Curves.

Mechanical synthesis of harmonic curves. Curve-tracing machines of Donkin and Cady. Kelvin's Tide-predicting Machine. Synthesis and analysis by Harmonic Analyzer of Michelson.

106. Composition of Parallel Vibrations of Same Period. A case of the composition of simple harmonic curves which is of great importance because of its application in theoretical optics is that in which the periods of the components are the same. Under these circumstances the resultant is a simple harmonic curve having the same period as that of the components. This will appear from the following consideration.

Writing $\omega_2=\omega_1$ the equation of the resultant curve becomes

$$y=r_1 \sin (\omega_1 x+\delta_1)+r_1 (\sin \omega_1 x+\delta_2),$$

whence, by expanding the values of $\sin (\omega_1 x+\delta_1)$ $\sin (\omega_1 x+\delta_2)$ we derive the equation

$$y=(r_1 \cos \delta_1+r_2 \cos \delta_2) \sin \omega_1 x+(r_1 \sin \delta_1+r_2 \sin \delta_2) \cos \omega_1 x.$$

In this equation we may write

$$r_3 \cos \delta_3=r_1 \cos \delta_1+r_2 \cos \delta_2, \quad . \quad . \quad . \quad . \quad (1)$$

and

$$r_3 \sin \delta_3=r_1+r_2 \sin \delta_2, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

subject to a subsequent determination of the values of r_3 and δ_3. Then

$$\begin{aligned} y&=r_3 \cos \delta_3 \sin \omega_1 x+r_3 \sin \delta_3 \cos \omega_1 x \\ &=r_3 (\cos \delta_3 \sin \omega_1 x+\sin \delta_3 \cos \omega_1 x) \text{ or } y=r_3 \sin (\omega_1 x+\delta_3). \end{aligned}$$

This represents a simple harmonic curve of period ω_1 amplitude r_3, and epoch δ_3.

The values of r_3 and δ_3 are easily found as follows:

By squaring Eqs. (1), (2), adding the results and simplifying we have

$$r_3^2=r_1^2+r_2^2+2r_1r_2 \cos (\delta_1-\delta_2),$$

whence the amplitude of the resultant is

$$r_3=\sqrt{r_1^2+r_2^2+2r_1r_2 \cos (\delta_1-\delta_2)}.$$

Dividing (2) by (1) we have $\tan \delta^3=\dfrac{r_1 \sin \delta_1+r_2 \sin \delta_2}{r_1 \cos \delta_1+r_2 \cos \delta_2}.$

The cases where the amplitudes r_1, r_2 are equal should be particularly noticed. If in addition to this equality $\delta_1-\delta_2=0$, $r_3=2r_1$; that is, if the components are in the same phase the amplitude of the resultant curve is double that of either component. If $\delta_1-\delta_2=\pi$, $r_3=0$; that is, if the components are in precisely opposite phases, the amplitude of the resultant $=0$ and the curve is reduced to a right line coinciding with the axis of X. For intermediate values of the phase-difference the amplitude varies between these two extremes.

It follows, furthermore, from what precedes that the addition of any number of simple harmonic curves of the same period gives as a resultant a simple harmonic curve also of the same period. For the components may be combined successively, each with the resultant of those previously taken.

107. Combination of Rectangular Harmonic Vibrations. Measuring the phase-angles from Y and X, respectively, the component vibration parallel to Y may be represented by the equation $y=r\cos\omega t$, that parallel to X by the equation $x=r_1\cos(\omega_1 t+\delta)$.

For purposes of illustration it will suffice to consider the simplest case, that in which the periods are the same and hence $\omega_1=\omega$. We have, therefore, $x=r_1\cos\delta\cos\omega t-r_1\sin\delta\sin\omega t$.
But

$$\cos\omega t=\frac{y}{r},\quad \sin\omega t=\sqrt{1-\frac{y^2}{r^2}}.$$

Hence

$$x=r_1\cos\delta\left(\frac{y}{r}\right)-r_1\sin\delta\left(\sqrt{1-\frac{y^2}{r^2}}\right),$$

and

$$x^2+r_1^2\cos^2\delta\left(\frac{y^2}{r^2}\right)-2r_1x\cos\delta\left(\frac{y}{r}\right)=r_1^2\sin^2\delta-r_1^2\sin^2\delta\left(\frac{y^2}{r^2}\right),$$

which may be written

$$\frac{x^2}{r_1^2}+\frac{y^2}{r^2}-\frac{2xy}{r_1r}\cos\delta=\sin^2\delta.$$

This is the equation of an ellipse.

If $\delta=0, \left(\frac{x}{r_1}-\frac{y}{r}\right)^2=0$ and $\frac{x}{r_1}=\frac{y}{r}$,

which is the equation of a straight line passing through the origin and making an angle $\tan^{-1}\left(\frac{r}{r_1}\right)$ with X.

If $\delta=\frac{1}{4}(2\pi)$, $\frac{x^2}{r_1^2}+\frac{y^2}{r^2}=1$, the equation of an ellipse with its axes lying in X and Y. If in addition $r=r_1$, $x^2+y^2=r^2$, and the ellipse becomes a circle.

If $\delta=\frac{1}{2}(2\pi)$, $\frac{x}{r_1}=-\frac{y}{r}$, the equation of a straight line through the origin and making an angle $\tan^{-1}\left(-\frac{r}{r_1}\right)$ with X.

If $\delta=\frac{3}{4}(2\pi)$, $\frac{x^2}{{r_1}^2}+\frac{y^2}{r^2}=1$, and if also $r=r_1$, $x^2+y^2=r^2$.

The ellipses represented by the preceding equations are always inscribed in a rectangle of which $2r$, $2r_1$ are the sides. The points of contact of the ellipse with the sides of this rectangle may be found by determining the intersections of the curve with the lines $x=\pm r$, $y=\pm r_1$.

If the periods of the vibrations to be combined have any other ratio, e.g., 1 : 2, or 1 : n, in the general equation we must make $\omega=2\omega_1$ or $\omega=n\omega_1$. But except in a few special cases the resulting curves are of very complex character and are more readily studied by graphical methods. (See text-book.)

If the difference of phase represented by δ varies gradually from 0 to 2π the curve will gradually pass in succession through all its corresponding variations. Fig. 26 illustrates these for the several ratios there indicated.

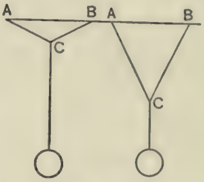

FIG. 25.

108. Applications. Blackburn's Pendulum (Fig. 25); Harmonograph Pendulums; Wheatstone's Kaleidophone; Lissajous' Curves (Fig. 26); Lissajous' Comparator; Mechanical Combination by Machines of Pickering, Donkin, and others.

109. Fourier's Theorem. “ If any arbitrary periodic curve be drawn, having a given wave-length, λ, the same curve may always be produced by compounding harmonic curves (in general infinite in number, having the same axis, and having λ, $\frac{1}{2}\lambda$, $\frac{1}{3}\lambda$, . . . for their wave-lengths.

“ The only limitations to the irregularity of the arbitrary curve are, first, that the ordinate must be always finite; and, secondly, that the projection on the axis of a point moving so as to describe the curve must move always in the same direction.

“ These conditions being satisfied, a wave of the curve may have any form whatever, including any number of straight portions.

“ Analytically the theroem may be expressed as follows:

“ It is possible to determine the constants C, C_1, C_2, etc., a_1,

a_2, etc., so that a wave of the periodic curve definied by the equation

$$y=C+C_1 \sin\left(\frac{2\pi x}{\lambda}+a_1\right)+C_2 \sin\left(2\frac{2\pi x}{\lambda}+a^2\right)+\ .\ .\ .$$

or

$$y=C+\sum_{i=1}^{i=\infty} C_i \sin\left(\frac{2i\pi x}{\lambda}+a_i\right),$$

shall have any proposed form, subject to the conditions mentioned above."

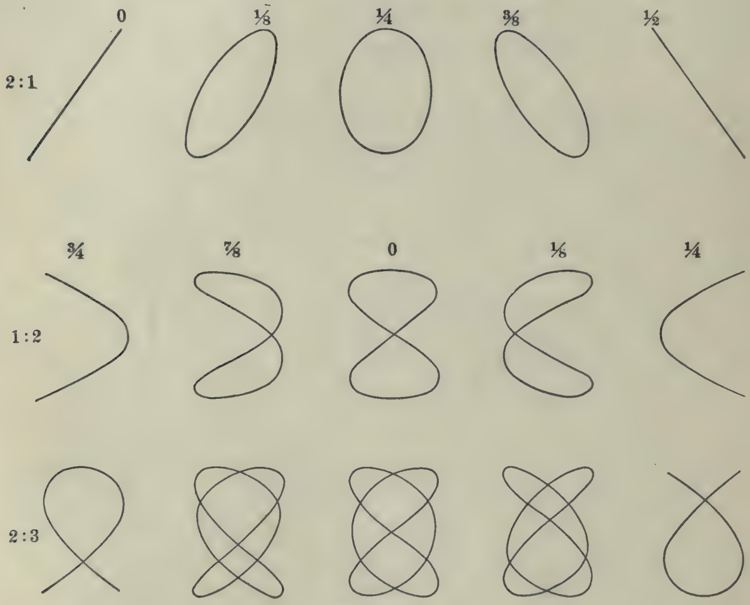

FIG. 26.

The preceding statement is extracted from Donkin's " Acoustics " (Clarendon Press Series).

110. Vibrations of Elastic Bodies. Since in the case of an elastic body subjected to any kind of deformation the restoring force within certain limits is proportional to the displacement, it follows that vibrations of small amplitude performed under the influence of elasticity are simple harmonic in character, and isochronous.

In case the amplitude of the vibration is so great that the restoring force is not at such point proportional to the first power of the displacement, the motion while still periodic is no longer simple harmonic in its character. It can be shown by analysis that under these circumstances the vibration is a compound harmonic one, represented by the resultant of a series of simple harmonic vibrations of frequencies n, $2n$, $3n$, etc., the particular harmonics present being determined by the law of variation of the restoring force.

Experimental evidence of these effects is clearly seen in the case of a vibrating tuning-fork. When gently bowed only the fundamental (frequency $= n$) can be heard, but with strong bowing the octave (frequency $= 2n$), the fifth of the octave (frequency $= 3n$), and even higher harmonics also appear.

A like occurrence, to be explained in a similar manner, is the occasional appearance of harmonic notes in telephonic transmission when a microphone transmitter is used.

The mathematical principles underlying these several phenomena are stated in § 109.

111. Acoustic Vibrations. The formula $T = 2\pi\sqrt{\frac{y}{a}} = 2\pi\sqrt{\frac{d}{a}}$ (in which d is the displacement) enables us to obtain the laws of vibration of bodies vibrating under the influence of elasticity.

The following demonstrations are of importance in acoustics, as giving the laws of vibration of sounding bodies.

Let $n = \frac{1}{T}$ be the frequency of vibration. Then $n = \frac{1}{2\pi}\sqrt{\frac{a}{d}} = \frac{1}{2\pi}\sqrt{\frac{F}{Md}}$, where F is the restoring force and M the mass in vibration.

For a bar or wire, $M =$ length × cross-section × density $= lsw$, and therefore $n \propto \sqrt{\frac{F}{lswd}}$.

For a bar or wire executing longitudinal vibrations, $F = \frac{Esd}{l}$, E being Young's modulus of elasticity; whence $n \propto \sqrt{\frac{Esd}{l^2swd}} \propto \frac{1}{l}\sqrt{\frac{E}{w}}$.

Hence for longitudinal vibrations the frequency is independent of the section of the bar, and varies inversely as the length, inversely as the square root of the density and directly as the square root of the modulus of elasticity of its material.

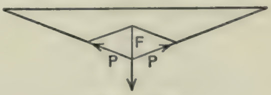

FIG. 27.

For an elastic string vibrating transversely, calling P the tension of the string when deflected, $F : P :: 2d : \frac{1}{2}l$. (See Fig. 27.)

Hence $F = \frac{4Pd}{l}$ and $n \propto \sqrt{\frac{4Pd}{l^2swd}} \propto \frac{1}{l}\sqrt{\frac{P}{sw}} \propto \frac{1}{lr}\sqrt{\frac{P}{w}}$, as $s = \pi r^2$.

Hence, for such a vibrating string the frequency varies inversely as the length of the string, directly as the square root of the tension, inversely as the diameter and inversely as the square root of the density.

For a vibrating elastic rod of length l, breadth b, depth k, $F=\frac{Abk^3d}{l^3}$, A being a constant. Hence $n \propto \sqrt{\frac{Abk^3d}{(lswd)l^3}} \propto \sqrt{\frac{bk^3}{(lbkw)l^3}} \propto \frac{k}{l^2}\sqrt{\frac{1}{w}}$. That is, for a bar vibrating transversely the frequency is independent of the breadth, inversely as the square of the length, directly as the depth, and inversely as the square root of the density.

These laws of variation of frequency are independent of the particular manner in which the bar is supported, only the absolute frequency being affected by this.

112. Circular Pendulum. The time of vibration of a circular pendulum swinging in an indefinitely small arc may be found as follows:

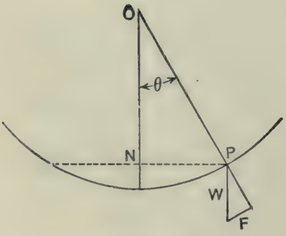

FIG. 28.

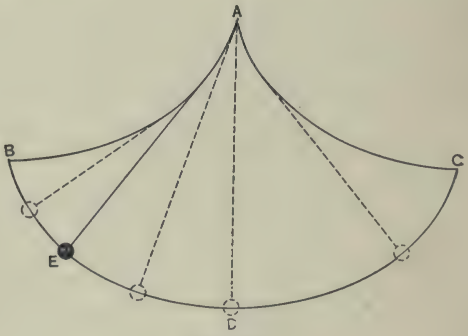

FIG. 29.

The vibration is simple harmonic since the accelerating component is $F=W \sin \theta = W\theta$ for small angles. (Fig. 28.) Let $OP=l$ and $PN=r$. Then $T=2t=2\pi\sqrt{\frac{r}{a}}=2\pi\sqrt{\frac{l \sin \theta}{g \sin \theta}}=2\pi\sqrt{\frac{l}{g}}$, and $t=\pi\sqrt{\frac{l}{g}}$.

113. Cycloidal Pendulum. The law of curvature of the cycloid such that with the cycloidal pendulum the accelerating (tangential) component of the weight of the vibrating particle ($W \sin \theta$) is always proportional to the arc of displacement. Hence the vibrations are isochronous for all arcs.

This property of the cycloid follows directly from the fact that with it the length of the arc included between the vertex and any point E (Fig. 29) is $d=4a \cos \frac{\phi}{2}$ where a is the radius of the generating circle and ϕ the angle made with the vertical by this radius drawn through E. Calling θ the angle of deviation when the vibrating particle is at E, $\frac{\phi}{2}=(90^\circ-\theta)$. The restoring force at E is $F=W \sin\theta$ $=\mathrm{W}\cos(90^\circ-\theta)=W\cos\frac{\phi}{2}$. Hence $F\propto\cos\frac{\phi}{2}\propto d$.

It will also be seen from the demonstration in § 112 that the time of vibration for all arcs is $t=\pi\sqrt{\frac{l}{g}}$.

113a. Torsion Pendulum. The time of a *single* vibration of a torsion pendulum is $t=\pi\sqrt{\frac{Tl}{K}}$, where l is the length of the wire suspension and K a constant.

On theoretical grounds, confirmed by experiment, we know that the angular displacement produced by a torque T applied at the free end of the wire is $\theta\propto Tl$ or $\theta=\frac{1}{K}Tl$, where K is a constant usually placed in the denominator, and sometimes called the coefficient of torsion. Substituting this value in the general formula

$$t=\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}},$$

we have

$$t=\pi\sqrt{\frac{\theta}{\alpha}}=\pi\sqrt{\frac{Tl}{K\alpha}}=\pi\sqrt{\frac{I\alpha l}{K\alpha}}\text{ as }T=I\alpha;\text{ i.e., }t=\pi\sqrt{\frac{Il}{K}}.$$

Hence (1) t is independent of the amplitude; that is, the vibrations are isochronous; (2) t is proportional to the square root of the moment of inertia of the system; (3) t is proportional to the square root of the length of the wire.

Propositions 1, 2, 3 are also found to be true by direct experiment. Hence, conversely, it follows that in torsional strain $\theta\propto Tl$. This is in fact the most satisfactory experimental manner of ascertaining the laws of torsion for wires as distinct from bars.

114. Damped Vibrations. When a vibrating body executes free vibrations the amplitude of these gradually diminishes owing to the energy expended in overcoming external or internal resistances. This effect is called *damping.*

In general the effect of the resistance is proportional to the velocity simply, and the diminution of amplitude is proportional to the amplitude itself at each instant. That is, denoting amplitudes by y and times by x $\frac{dy}{dx}=-ky$, whence $\frac{dy}{y}=-kdx$. Hence, by integration

$$\log y=-kx+\text{constant and }y=ce^{-kx}.$$

It follows from this equation that if we designate by y_m and y_n respectively the amplitudes of the decreasing vibration after corresponding times x_m, x_n, and, if during the interval $x_n - x_m$, N vibrations have been made, then $\frac{\log y_m - \log y_n}{N} = \Lambda$, a constant, which is known as the *logarithmic decrement* of the vibration, and is evidently the decrement per vibration of the logarithm of the amplitude.

It will also be seen that the vibration of a particle subject to damping may be represented graphically by replacing the reference circle in Fig. 24 by a logarithmic spiral.

The periodic curve representing a damped vibration must be characterized by a diminution of its ordinates from those of the sine curve representing the corresponding displacement with an undamped vibration of the same period, which follows the law represented by the equations just demonstrated. Hence the equation of such a harmonic curve will take the form $y = e^{-kx} r \sin(\omega x + \delta)$.

Fig. 30 illustrates the method of constructing the curve representing a damped simple harmonic vibration. The abscissas are made proportional to times and hence to the angle θ. The ordinates are made proportional to the value for the angle θ of the ordinate of the extremity of the radius vector of the logarithmic spiral of reference, i.e., to $r_\theta \sin \theta$.

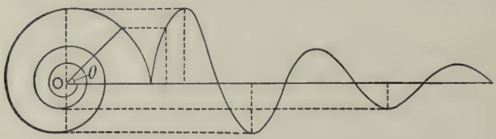

FIG. 30.

115. Stroboscopic Method of Studying Periodic Motion. When a rapidly moving body is illuminated by a flash of light of such brief duration that the body does not traverse a perceptible space during its continuance, the object appears to be at rest. This is the case, for example, with a swiftly revolving wheel illuminated by the electric spark. If there is a series of brief illuminations the object is seen in a corresponding series of positions. If a body possessing a periodic motion is illuminated by a series of recurrent flashes, and if the interval between each flash is such as to coincide exactly with that period (or with n times the period) the body will appear to be absolutely at rest, however great its velocity may be. It will be seen continuously owing to the persistence of vision. If the interval between the flashes is slightly less than the periodic time the body will apparently possess a slow retrograde motion. If the interval is slightly greater than the periodic time, there will be an apparent slow forward motion of the body. In this way changes of configuration or of form may be studied.

The flashes may be obtained by any convenient method, as by the electric spark or by throwing the light from an electric arc through a rotating disk with radial slits; or the moving object while continuously illuminated may be viewed directly by the eye, looking through a rotating slotted disk.

116. Illustrations and Applications. (*a*) *Single flash.* Flashlight photography. Rapid-shutter photography. Study of trotting horses, etc., by Muybridge. Spark-photography of flying bullets by Mach and Boys.

(*b*) *Stroboscope.* Determination of period of rotating wheel. Study of motion of chain of small bones in human ear by Mach. Determination of frequency of tuning-fork. Stroboscopic study of alternating arc in Rogers Laboratory, M. I. T. Use of microstroboscope in telephonic studies in Rogers Laboratory.

Toys; e.g., Phenakistiscope of Plateau. Stroboscopic disks of Stampfer. Dedaleum of Horner.

(*c*) *Kinetoscope; Kinematograph.* Early application of photography by Muybridge to study of gait and postures of animals. Representation of movement of express train, of mountain ascents, etc. Representation of progressive motion of waves by Müller and Wood. Study of plant growth.

117. Free and Forced Vibrations. If a body so conditioned as to be capable of entering into a state of vibration is subjected to the action of a disturbing impulse and then left free, it will execute what are called *free vibrations*, the period of these being determined by the mass, form, and mechanical conditions of the body. If such a body is subjected to the continued action of a periodically varying force, external to itself, this will act to produce in the body vibrations of the same period as that of the force variation. These are called *forced vibrations.*

If the varying force coincides in period with any vibration which it is possible for the body to execute, extremely strong forced vibrations may be produced, even though the effect of each separate periodic action of the force is small. In this case vibrations thus excited are often called *sympathetic vibrations* or *syntonic vibrations.*

When the action of the force is reciprocating, so that periodic reversals in its direction occur, it can only produce strong sympathetic vibrations in a body which is capable of vibrating in the same period.

But if the force acts intermittently, producing a series of equally timed impulses, it may excite sympathetic vibrations in a body which is capable of vibration in any period which is an integral number of times that of the impulses.

Illustrations of the former of these conditions are found in the acoustic phenomena of the sympathetic tuning-forks and resonance.

The latter conditions are found in many cases of forced mechanical vibrations, as in the ringing of a church bell, in Galileo's experiment in which a heavy pendulum is set into vibration by properly timed puffs of the breath, in the vibrations of suspension and other bridges excited by marching bodies of troops, and in the vibration of mill buildings by machinery.

118. Applications. Hardy's " Noddy " used in pendulum experiments. Frahm's Resonance Apparatus. Hartmann and Braun's Frequency Meter for Alternating Currents. Helmholtz's Harmonic Tuning Forks. Clocks governed by action of electro-magnet upon Pendulum.

119. Phase Relation. There is always a certain difference of phase between the force which acts to produce the forced vibration and the resulting forced vibration itself, the latter lagging behind the former. The amount of this difference depends upon the relation between the periods of the force and the free vibration. If these are precisely the same, the case of perfect unison, the phase-difference can be shown to be $\frac{1}{2}\pi$, that is, one-fourth of a period. If the natural frequency is the greater, the lag of the forced vibration is less than a quarter period; if the natural frequency is the lesser, the lag is between one-quarter and one-half period.

Illustration. Phenomena observed with Sympathetic Pendulums.

120. Coexistence of Free and Forced Vibrations. When a body is set into forced vibration by the action upon it of a periodic force there is often noticed at the beginning of the operation, before the forced vibration is fully established, a rhythmic variation in the amplitude of the vibration actually performed. In the case of an acoustic vibration " beats " are heard.

This phenomenon is caused by the simultaneous existence of a free vibration, due directly to the disturbance of the vibratory body and the forced vibration impressed upon the body. If these two periods are not exactly alike beating must occur.

Illustrations. Beats of tuning-fork electrically driven by a governing fork of same frequency. Beats of electrically excited musical string. Frahm's Resonance Top.

121. Resonance. In acoustics the term resonance is usually applied to the phenomena of forced vibrations, especially when they are produced in a mass of air.

The small density of air enables it to enter into a state of forced vibration very readily even when the period of this is considerably removed from the period of free vibration, although the resonance rises rapidly in amount as the period of the forced vibration approaches that of the free vibration of the mass of air.

Any mass of air or other body which is capable of entering into a state of vibration will respond by resonance to sounds having the same pitch (and hence the same vibration-frequency) as any which it can emit in consequence of its own free vibration.

Illustrations of Resonance of Mass of Air. Tuned Bottles. Helmholtz's resonators. Open and stopped organ pipes.

In case a body is capable of vibration in several different periods it will respond to any one of the several notes of corresponding vibration-frequency. It will also respond to several such sounds simultaneously.

Illustrations. Multiple resonance of organ pipes. Simultaneous resonance of organ pipe or two or more forks.

122. Conditions Affecting Damping. The more readily a body enters into a state of forced vibration, the more readily are its vibrations damped. For example, a mass of air in a resonator can be forced by a powerful tuning-fork to vibrate in a period far removed from its natural rate of free vibration; but the sound of such a vibrating mass of air persists only for a very brief period. On the other hand, a tuning-fork can only be made to execute forced vibrations when the exciting fork is very closely in unison with it; but vibrations excited in any manner in the fork are very persistent.

Whenever a vibrating body causes sympathetic vibrations in a second body, the former loses its own energy of vibration more rapidly in proportion to the strength of the resonance which it causes. Hence, the damping of the vibrations of a body in proximity to others capable of vibrating at the same rate as itself will be more rapid than if it is removed from the neighborhood of such bodies. The reaction of the sympathetically vibrating body upon the other is always such as to oppose its motion.

Interference of Waves

123. Phenomena. If two trains of parallel waves of equal length meet one another and coalesce, the waves constituting the resultant train will possess an amplitude equal to the algebraic sum of the amplitudes of the components. The wave length will remain unchanged. If the waves are in similar phase the resultant wave will be of greater amplitude than either component; if they are in unlike phase, less. In the particular case where the amplitudes of the coalescing waves are equal and their phase opposite they will neutralize each other and the amplitude of the resultant wave will be zero. This effect is known as *interference.* If the amplitudes of the waves thus compounded are unequal, the interference will be only partial.

This phenomenon is considered analytically in § 106.

In most cases of interference the interfering trains of waves start from the same source and reach the place of coalescence by two different paths, one of which is half a wave length or an odd number of half wave lengths longer than the other. Under these circumstances the two sets of waves always meet trough to crest and so interfere, completely or partially, according to their relative amplitudes.

In the case of sound waves meeting in opposite phase, i.e., condensation to rarefaction, the addition of one sound to another of the same pitch and loudness, and hence of the same wave length and amplitude, may produce silence. In like manner the addition of light to light may produce darkness.

Illustrations. Water Waves. Absence of tide at Batsha, where the tidal wave reaches the port by two channels of different length. Increased tide at certain points on eastern coast of England arising from interference of wave trains traveling respectively northward from English Channel and southward from sea above Scotland.

Waves in cords. Stationary waves in Melde's experiment.

Sound waves. Herschel's trombone apparatus. Stationary waves formed in free air by coalescence of trains of waves, respectively direct and reflected from a wall.

Light waves. Colors of thin films, as in case of soap bubble.

123a. Periodic Interference; Acoustic Beats. When two notes approximately but not exactly in unison are sounded simultaneously, a periodic increase and decrease in the loudness of the sound results, due to alternate reinforcement and interference of the sound waves. *The frequency of the beats is equal to the difference in frequency of vibration of the two notes producing them.*

Gravitation

124. Law of Universal Gravitation. *Every particle of matter in the universe attracts every other particle with a force varying directly as the product of their masses, and inversely as the square of their distance.*

Hence the mutual gravitation of two masses m, m', at a distance d, is

$$G=\phi\frac{mm'}{d^2}.$$

The general theory of gravitation, as proved by Newton during the years 1666 to 1687, followed from his establishment of the following facts:

" 1. That the force by which the *different* planets are attracted to the sun is in the inverse proportion to the squares of their distances. [1. Results from Third Law of Kepler.]

" 2. That the force by which the *same* planet is attracted to the sun, in different parts of its orbit, is also in the inverse proportion to the square of the distance. [2. Results from First and Second Laws of Kepler.]

" 3. That the earth also exerts a force on the *moon*, and this force is identical with the force of *gravity*.

" 4. That bodies thus act on *other* bodies besides those which revolve around them; thus the sun exerts such a force on the moon and satellites, and the planets exert such forces on *one another*.

" 5. That this force thus exerted by the general masses of the sun, earth, and planets arises from the attraction *of each particle* of these masses; which attraction follows the above law, and belongs to all matter alike."—WHEWELL.

The laws of Kepler referred to above were determined by that astronomer wholly from observation (1609; 1619). They are as follows:

1. *The planets all revolve in ellipses with the sun in one focus.*
2. *The radius vector describes areas proportional to the times.*
3. *The squares of the periodic times of the planets are proportional to the cubes of their mean distances from the sun.*

125. Value of Gravitation Constant. The constant ϕ in the formula for gravitation denotes the attraction existing between two masses of 1 g. each at a distance of 1 cm. apart. Its value has been determined by the method of Cavendish (1798) in which a small metallic sphere carried on the arm of a torsion balance is attracted by a heavy lead sphere. The force between the two spheres is known from the torsion developed in the wire of the balance. Their masses and distance apart are also known, so that the value of ϕ can easily be computed. Results by Boys, who used a quartz fiber suspension (1895) give as its value $\phi=6.6576\times10^{-8}$. By comparing the attraction of the lead sphere on the sphere of the torsion balance with the attraction of the earth on the same, the mean density of the earth can be determined. The value corresponding to the value of ϕ given above is $\Delta = 5.5270$.

Values of ϕ agreeing closely with that given above have also been obtained by other methods. Thus Poynting (1891) measured the attraction between two lead spheres by means of a chemical balance of great sensitiveness.

The law of gravitation appears to hold throughout all distances, varying from interplanetary spaces certainly to within a few centimeters and presumably until the attracting bodies are in contact. ϕ is independent of the material of the masses, and of the medium separating them. Also gravitation is in no case directive or polar in character.

126. Diminution of Gravity without Surface of Sphere. Newton showed that the resultant effect of all the particles of matter composing a homogeneous sphere upon a particle outside of it is the same as if the total mass of the sphere were concentrated at its center, Hence, calling G, G_h, the respective attractions upon a body at the surface of a sphere of radius R, and at a distance h above the surface,

$$G : G_h :: \frac{1}{R^2} : \frac{1}{(R+h)^2};$$

whence

$$G_h = G\frac{R^2}{(R+h)^2},$$

or

$$G_h = G\left(1 - \frac{2h}{R}\right), \text{ approximately.}$$

The preceding formula can be applied to the diminution of gravity on ascending above the surface of the earth, where G, G_h are the weights of the body. Also, it follows that

$$g_h = g\left(1 - \frac{2h}{R}\right), \text{ approximately.}$$

Von Jolly (1881) made a direct determination of the diminution of g in ascending, by the use of an equal-arm balance with two sets of pans, one close to the beam, the other at a distance below it, and, consequently, nearer the surface of the earth. With a distance of 21 m. between the pans, a mass of 5 kg. weighed 32 mgm. more when in the lower than when in the upper pan.

Von Jolly furthermore found that by placing a sphere of lead of mass 5775 kg. immediately under the movable mass when thus lowered, the weight of the latter was increased by one-half of a milli-

VARIATION OF g WITH ELEVATION

	Elevation ft.	g cm.	
San Francisco	375	979.951	Mendenhall
Lick Observatory, Mt. Hamilton	4205	979.646	"
Denver	5374	979.595	Putnam
Pike's Peak	14085	978.940	"
Kawaihae	8	978.798	Preston
Kalaieha	6660	978.485	"
Waiau, Summit Mauna Kea	13060	978.055	"
Do. Reduced to Lat. of Kawaihae	.....	978.067	"
Tokio	0	979.84	Mendenhall
Fuji, Summit	12441	978.86	"
Do. Reduced to Lat. Tokio	.....	978.65	"

gram. This was due to the gravitational attraction between the two spheres. It follows directly from this that the earth's attraction on the movable mass was 10,000,000 times that of the lead sphere, and, hence, that the mean density of the earth should be about 5.7.

Richarz and Krigar-Menzel using a modification of this method obtained a value of $\Delta = 5.5$, agreeing very closely with that obtained by Boys by the torsion balance.

In a more general form this theorem is true for all forces for which the law of inverse squares holds. It will be preferable to give the demonstration in connection with the discussion of electrical attraction and repulsion.

127. Instruments for Direct Measure of Variations in Gravity. Mass supported by spring; Siemens' bathometer. Threlfall and Pollock's quartz-thread gravity balance. Von Sterneck's barymeter; oblique column of mercury balanced on knife-edge support.

128. Effect of Spheroidal Form of Earth. Analysis and observation both show that the force of gravity increases from the equator to the poles. The actual gain of weight of a body carried from the former to the latter latitude would be about $\frac{1}{189}$ of its original value. Of this approximately $\frac{1}{288}$ arises from the centrifugal force caused by the rotation of the earth. The remainder, about $\frac{1}{545}$, is due to the fact that the earth is an oblate spheroid.

At the poles $g = 983.194$ cm.
In latitude 45° $g = 980.630$ cm.
At the equator $g = 978.066$ cm.

Value of g_0 for any place.—If g_0 be the value of the acceleration of gravity in latitude 0° and at sea-level, the acceleration g_λ in latitude λ and at altitude h will be $g_\lambda = g_0\ (1+0.005243 \sin^2 \lambda) \left(1-\frac{2h}{R}\right)$.

This is the value which g_λ would have in mid-air. In the actual case of stations in mountainous regions various particular formulæ have been used, as no general formula can be deduced which will take all local disturbances into account. (See §150 for a further consideration of the subject.)

129. Gravity within Sphere. It can be shown that within a homogeneous sphere the weight of a mass would be directly proportional to its distance from the center. But this condition is very far from being true of the earth owing to the increased density of deeper strata. In fact, g at first increases with increasing depth. Below a certain but unknown depth it must obviously decrease.

129a. Field of Force. Potential. The consideration of certain branches of Physics is greatly facilitated by the use of the conceptions of the *field of force* and *potential.* The general principles of this treatment may conveniently be considered in this place. The following statements refer particularly to gravitational force, but it will be seen later that the same conceptions are employed in the discussion of electricity and magnetism.

A *field of mechanical force* is any region of space within which force acts. A line whose direction at every point is that of the resultant force at that point is called a *line of force.* The magnitude of the force, due to the field, acting upon a unit of mass (gram) measures the *strength of the field.*

In every field of force a condition exists, in virtue of which a mass tends to move from any one position to some other under the action of force. It is usual to consider the condition of the field in this regard by reference to what is known as the *potential.* A mass tends to move from a point of higher to one of lower potential. The potential of the gravitational field due to the earth increases outwardly with the distance from the surface. It will be seen from the preceding statements that gravitational potential is analogous to gravitational level.

The potential difference between two points in a gravitational field is numerically measured by the work done in moving a gram (overcoming its weight) from one point to the other; that is, from the one at lower to the one at higher potential. The potential is usually reckoned conventionally as increasing in the direction in which *positive work* is done.

In mathematical physics the *absolute gravitational potential* at a point may be defined as the work done in moving a gram from that point to an infinite distance.

In general we are concerned only with differences of potential.

The difference of potential between any point at an elevation h and the surface of the earth, assuming g to be constant, is gh. This is evidently the mutual potential energy of the earth and a gram located at that point.

If a mass be so carried in a field of force as always to move at right angles to the lines of force, no work, either positive or negative, will be done. Any surface within which such motion is possible is an *equipotential surface.* An equipotential surface must at every point be at right angles to the line of force passing through that point. *A level* surface fulfills that condition.

It follows from the definition in § 39 that the *force* in any direction acting upon a unit of mass *is equal to the rate of change of potential* in that direction. That is, if V is the potential, $F=\frac{dV}{dS}$.

130. Falling Bodies. Laws.

I. *The acceleration produced by gravity is independent of the mass*; whence *the velocity of a freely falling body is independent of its mass.*

Experimental Proof. Cannon-balls of different masses dropped from tower. (Galileo, about 1590, at Pisa.)

Theoretical Proof. Calling M, M' the masses of two bodies, G, G' their attraction to the earth, a, a' the accelerations produced, we have

$$G : G' :: M : M',$$

but

$$Ma : M'a' :: G : G';$$

whence

$$a=a'.$$

II. *The acceleration is independent of the material of the body.*

Experimental Proofs. (*a*) Falling balls of metal and cork; (*b*) guinea and feather tube; (*c*) proofs of Newton and Bessel by pendulum, to be discussed later.

III. *Velocity is proportional to the time of descent.*

IV. *Velocity is proportional to the square root of distance fallen through.*

V. *Space described is proportional to the square of the time.*

III, IV, V, follow from the fact that the motion is uniformly accelerated.

131. Verification of Laws by Experiment. The three last-mentioned laws of falling bodies may be investigated experimentally by means of various devices invented for the study of uniformly variable motion, including that produced by gravity. Of these the following are particularly worthy of mention:

1. *Inclined Plane* (Galileo). Apply preceding formulæ. (See § 26.)

2. *Atwood's Machine* (1780) (Fig. 31). The equal large masses, M, M are connected by a flexible cord running over a pulley A. The addition of a small mass m produces a uniformly accelerated motion, in which the acceleration is found from the equation

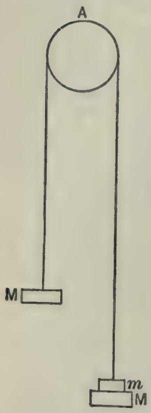

FIG. 31.

$$w=mg=a(2M+m),$$

whence

$$a=g\frac{m}{2M+m}.$$

This value of a being substituted in the general formulæ, we have the velocity acquired by the mass and the space traversed by it, which may be compared with the results of experiment.

The effect of the pulley A must also be allowed for. This may be computed from its moment of inertia, but it is best determined by experiment. It will be the same as if the masses M, M were increased by a constant amount M'. Making this correction,

$$a=g\frac{m}{2M+M'+m}.$$

The following demonstration of the value of a with Atwood's machine is based on the principles of Energy.

If the heavier mass descends through a distance h and the system acquires thereby a velocity v, then $wh=mgh=\frac{1}{2}(2M+m)v^2$ $=\frac{1}{2}(2M+m)(2ah)$, observing that $v^2=2ah$, since the motion is uniformly accelerated. Solving this equation relatively to a, we have $a=g\frac{m}{2M+m}$, as before.

3. *Barbouze's Machine.* In this machine, a lamp-blacked cylinder is attached to the axis of the pulley of an Atwood's machine. Against this rubs a style attached to a tuning-fork, which marks equal intervals of time by its vibrations, and from the relative length of the sinuosities produced with increasing velocity, the laws can be determined experimentally.

A simpler method consists in causing a freely falling glass plate, covered with lamp-black, to press very lightly against a style carried by one of the prongs of a vibrating tuning-fork. In a later form of the apparatus the fork falls while the glass plate is fixed.

4. *Morin's Machine* (Fig. 32). A freely falling body traces its path against a cylinder moving uniformly about a vertical axis. By the combination of the two motions a parabola is traced, showing that the motion of the falling body is uniformly accelerated.

In a modified form of the machine, devised by Sir George Darwin, the pencil is fixed and the revolving cylinder falls.

5. *Additional Methods.* The laws of falling bodies are also studied by determining the velocity acquired and space traversed, by means of electric chronographs.

It is evident that any of these devices will enable us to obtain a value of g, of approximate correctness. Accurate methods of determining this will be explained in the chapter on the pendulum.

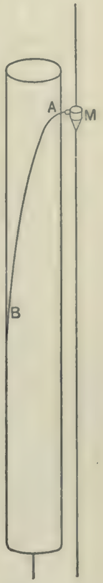

FIG. 32.

132. Case of Body Falling from Great Height. Since gravity varies inversely as the square of the distance from the center of the earth, in the case of a body falling through a great distance, this must be taken into account. For small distances the diminution may be neglected, as at the height of a kilometer it is only $\frac{1}{3188}$ of the value at the surface. It can be shown by analysis that the velocity acquired by a body falling freely to the earth from an infinite distance would be about 35,000 ft. per second.

Conversely, a mass projected vertically upward with this velocity would not return to the earth.

This principle has served to disprove an early hypothesis as to the lunar volcanic origin of meteorites, and also, by an application of the Kinetic Theory to indicate the probable constitution of planetary atmospheres.

133. Projectile. The trajectory of an unresisted projectile is a parabola.

Let OR (Fig. 33) be the original direction of projection and V_0 the initial velocity. Call the angle of elevation ROX, θ.

Let x, y, be the coordinates of any point P on the curve and t the time taken by the projectile to reach that point. Then

$$x=(V_0 \cos\theta)t, \quad \dots\dots\dots \quad (1)$$

$$y=(V_0 \sin\theta)t-\tfrac{1}{2}gt^2. \quad \dots\dots\dots \quad (2)$$

By eliminating t we have

$$y = x \tan \theta - \frac{g}{2V_0^2 \cos^2 \theta} x^2, \quad \ldots \ldots \quad (3)$$

which is the equation of the trajectory.

This is the equation of a parabola with a vertical axis.

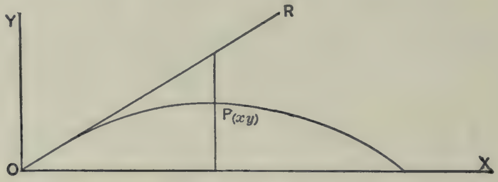

Fig. 33.

From (1), (2), (3), can be found the value of the horizontal and vertical ranges, the time of flight and the elevation for maximum horizontal range.

134. Effects of Air Resistance. The resistance of the air causes a large deviation from the parabolic trajectory and a great diminution of range in the case of rapidly moving projectiles. The maximum horizontal range is theoretically obtained when the angle of elevation is 45°, but this is practically true only for low velocities. For swift projectiles this angle is about 35°.

Other effects of air-pressure on a projectile are (*a*) gyroscopic deviation (drift) with elongated projectile from rifled gun, since the pressure tends to tilt the top of the projectile upward; (*b*) lateral deviation from the plane of projection in case of a ball rotating about a vertical axis, caused by unequal air-pressure on opposite vertical halves according as the velocity at the surface due to rotation is with or against the motion of translation; as in the case of curve pitching in baseball.

In the case of an unsymmetrical rotating projectile the air-resistance may give rise to a very complex path, as, for example, in the Australian *boomerang*, a curved club which returns to the place from which it is thrown.

135. Velocity Acquired in Descending Frictionless Inclined Plane. Since $v = \sqrt{2gs\frac{H}{L}}$, if $s = L$, that is, if the body traverses the whole length of the plane, $v = \sqrt{2gH}$. This is independent of L, the length of the plane, and is equal to the velocity acquired by a freely falling body in descending through the vertical height H.

136. General Proposition. *A body descending from a given point to a given horizontal plane will acquire the same velocity whether this descent is made vertically or obliquely over an inclined plane or over a curved surface.* This proposition is, of course, departed from in practice, because of the effect of friction and other resistances.

137. Properties of Cycloid. The path along which a body will descend most swiftly, between two points not in the same vertical, is an inverted cycloid passing through these points with its cusp at the uppermost of them. For this reason, the cylcoid is often called the *Brachystochrome.*

Another important property of the cylcoid is, that the time required to descend to the lowest point of the complete inverted curve is the same, from whatever point of the curve the body may start. The cycloid is, therefore, a *Tautochrome.*

Pendulum

138. Case of Body Rolling on Curve or Suspended by Flexible Cord. See Figs. 34, 35.

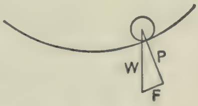

FIG. 34.

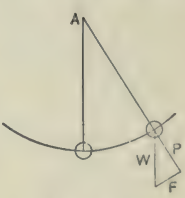

FIG. 35.

139. Simple or Mathematical Pendulum. This may be defined as *a gravitating particle suspended by a cord without weight.*

140. Time of Vibration. For very small circular arcs $t=\pi\sqrt{\frac{l}{g}}$. (See §112.) This is independent of amplitude. (Galileo, about 1583.)

For all circular arcs,

$$t=\pi\sqrt{\frac{l}{g}}\left[1+\left(\frac{1}{2}\right)^2\left(\frac{h}{2l}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{h}{2l}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{h}{2l}\right)^3+\ .\ .\ .\ \right],\quad \text{or}\quad t=\pi\sqrt{\frac{l}{g}}\left[1+\frac{1}{4}\cdot\frac{h}{2l}\right],$$

a very close approximation used in practice. In the formula h is the "sagitta" of the arc of vibration.

141. Isochronism. If the particle moves in a cycloidal arc, the formula $t=\pi\sqrt{\frac{l}{g}}$ is true for all amplitudes. (Huygens, 1673.) (See §113, for proof.)

Hence, for a cycloidal pendulum, or approximately for a circular pendulum vibrating through small arcs, the vibrations are isochronous; that is, performed in equal times, independently of the amplitude.

Construction of cylcoidal pendulum. (See Fig. 36.)

142. Laws of Pendulum.

I. *Time is independent of amplitude* (subject to limitations stated above).

II. *Time varies as the square root of length.*

III. *Time varies inversely as the square root of g.*

IV. *Time is independent of mass or material of pendulum.*

143. Physical Pendulum. The time of vibration is varied by changes in the distribution of the mass of the pendulum.

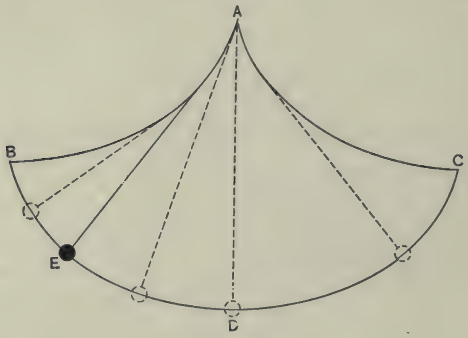

Fig. 36.

144. Center of Oscillation. This is that point of a physical pendulum which vibrates in the same time as if it were free from all connection with the remaining particles. Its position can be determined mathematically for homogeneous bodies of regular form. In a prismatic rod suspended at one end, it is at a distance of two-thirds the length of the rod from the point of suspension. The *length* of a physical pendulum is the length of the *equivalent simple pendulum*, and is equal to the distance from the axis of suspension to the center of oscillation.

It can be shown that for any physical pendulum this is equal to the moment of inertia of the system relatively to the axis of suspension, divided by the product of the mass of the system into the distance from the axis of suspension to the center of gravity. (See § 145.)

The center of oscillation can also be shown to be the *center of percussion*, which is the point at which a body suspended from an axis may be struck a blow in its plane of rotation without producing any pressure upon the axis.

The center of oscillation and axis of suspension are mutually convertible; that is, the time of oscillation is the same from whichever of these points the pendulum is suspended. (See §146 for demonstration.)

These properties of the physical pendulum were discovered by Huygens and announced in 1673.

145. Length of Equivalent Simple Pendulum. The length of the simple pendulum whose time of vibration is the same as that of a particular physical pendulum may be found as follows:

Call M the mass of the given physical pendulum, and I_p its moment of inertia relative to the point of suspension P (Fig. 37). Let G be its center of gravity and O its center of oscillation when suspended from P. The distance PO is the required length, l. With the *equivalent simple pendulum* the mass M will be concentrated at O. Let $PG=d$.

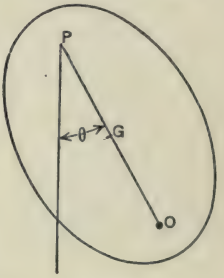

Fig. 37.

It is necessary and sufficient for equality in the periods of the physical and equivalent simple pendulum that for every value of the angle of deflection θ the angular acceleration α due to the torque produced by the weight of the pendulum shall be the same for both.

In general (§89) $Tt=I\omega$ whence $Tdt=Id\omega$, and $T=I\dfrac{d\omega}{dt}=I\alpha$.

For the physical pendulum the moment of the weight, acting through G at the instant when the deflection is θ, is $T_p=Mgd\sin\theta=I_p\alpha_p$, and for the simple pendulum, $T_s=Mgl\sin\theta=I_s\,\alpha_s=Ml^2\alpha_s$, whence

$\alpha_p=\dfrac{Mgd\sin\theta}{I_p}$, $\alpha_s=\dfrac{Mgl\sin\theta}{Ml^2}$. For equality of period of vibration $\alpha_p=\alpha_s$, whence $l=\dfrac{I_p}{Md}$.

The following examples will illustrate the application of the preceding principles:

1. Position of center of oscillation of prismatic rod of length L, suspended from one end.

$$I_g=\frac{1}{12}ML^2.\quad I_p=I_g+M(\tfrac{1}{2}L)^2=\tfrac{1}{3}ML^2;\ l=\frac{I_p}{Md}=\frac{\tfrac{1}{3}ML^2}{M\tfrac{1}{2}L}=\tfrac{2}{3}L.$$

2. Position of center of oscillation of sphere suspended by very fine wire of length h.

$$I_g=\tfrac{2}{5}Mr^2.\quad I_p=I_g+M(h+r)^2;\ l=\frac{I_p}{M(h+r)}=h+r+\frac{2}{5}\,\frac{r^2}{h+r}.$$

146. Convertibility of Point of Suspension and Center of Oscillation. That the time of vibration is the same from whichever point, P or O (Fig. 38) the vibrating mass is suspended follows from the fact that the length of the *equivalent simple pendulum* is the same in either case. Denote by k_g, k_p, k_o, the radii of gyration of the mass relatively to G, P, and O, respectively. The length of the equivalent simple pendulum when P is the point of suspension is

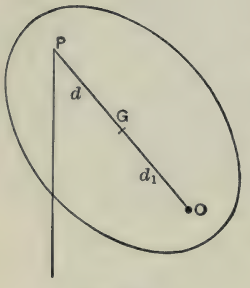

FIG. 38.

$$l_p=\frac{I_p}{Md}=\frac{Mk_p^2}{Md}=\frac{Mk_g^2+Md^2}{Md}=\frac{k_g^2}{d}+d. \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

When O is the point of suspension

$$l_o=\frac{I_o}{Md_1}=\frac{Mk_o^2}{Md_1}=\frac{Mk_g^2+Md_1^2}{Md_1}=\frac{k_g^2}{d_1}+d_1. \quad . \quad . \quad . \quad . \qquad (2)$$

Since $l_p=d+d_1$ we have from (1) $k_g^2=dd_1$.

Hence

$$l_0=\frac{k_g^2}{d_1}+d_1=\frac{dd_1}{d_1}+d_1=d+d_1=l_p;$$

that is, whether the vibrating mass is suspended from P or from O the length of the equivalent simple pendulum is the same, and hence the time of vibration is the same.

147. Metronome Pendulum. A pendulum in which a considerable portion of the mass is situated above the axis of suspension is called a *metronome pendulum*. By varying the position of a movable mass M' (Fig. 39), the time of oscillation and length of the *equivalent simple pendulum* can be varied within very wide limits, without increasing the dimensions of the apparatus. The metronome of Mælzel (1816) commonly used for giving time in music, is constructed on this principle, and is obviously far more convenient than the ordinary "bullet" pendulum previously employed.

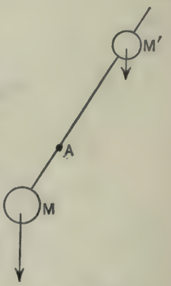

FIG. 39.

It is evident that with this form of pendulum the center of oscillation lies outside of and below the mass of the pendulum itself. Its position can be calculated by means of the formula in §145.

148. Determination of Length of Pendulum beating Seconds.—I. *Borda's Method.* (1790.)

Invariable Pendulum. Ball of platinum B (Fig. 41) is suspended by wire so light that it is without sensible influence on time of vibration. Length of wire (h) and radius of ball (r) are measured.

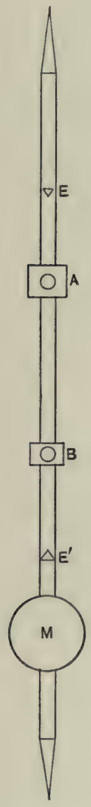

FIG. 40.

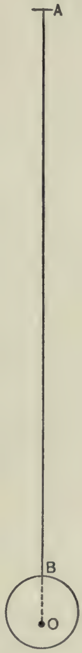

FIG. 41.

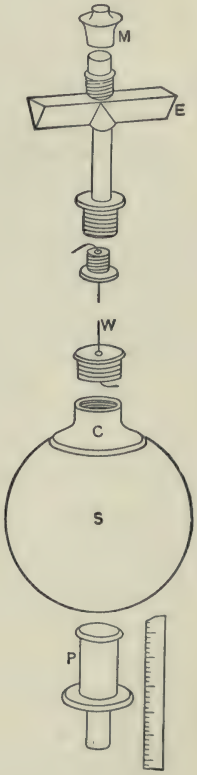

FIG. 42.

Distance of center of oscillation O below center of sphere is given by analysis. It is $x=\frac{2}{5}\frac{r^2}{h+r}$; the length of the equivalent simple pendulum is, therefore, $l=h+r+\frac{2}{5}\frac{r^2}{h+r}$.

The time t is determined by comparison with a clock. Then, calling l the distance, AO, l_s the length of the seconds pendulum,

$$t : 1 :: \sqrt{l} : \sqrt{l_s}.$$

The time t is determined by the method of coincidences.

Fig. 42 shows certain details of the apparatus.

II. *Kater's Method.* (1818.) *Reversible Pendulum.* The positions of the weights A, B (Fig. 40) are varied until the pendulum vibrates in the same time whether hung upon E or E'. Hence, when E is the axis of suspension, E' is the corresponding center of oscillation. The length of the equivalent simple pendulum is, therefore, the distance between E and E'. The time of vibration is measured by a clock, and the length of the seconds pendulum determined as in I.

With a reversible pendulum, if the time of vibration is approximately but not exactly, the same when suspended from either knife-edge, which is in reality the usual case in practice, it can be shown that the length of the equivalent simple pendulum is

$$l = \frac{(d_1+d_2)(d_1-d_2)}{d_1t_1^2 - d_2t_2^2},$$

when t_1, t_2 are the times of vibration, and d_1, d_2 the distances of the center of gravity from the two knife-edges.

Bessel (1826) used a modification of Borda's method. In this the length of the wire suspension is varied from d to d', and the corresponding times of vibration t, t' are determined. Then

$$l - l' = (d-d')\left(1 - \frac{2}{5}\frac{r^2}{dd'}\right), \text{ and } \sqrt{l} : \sqrt{l'} :: t : t'.$$

Hence we can determine l and l'. d, d' = distances from axis of suspension to center of sphere. The practical advantage is that $d-d'$ is the principal term, a quantity more easily measured than either d or d'. (See Fig. 43, in which $AA' = d-d'$.)

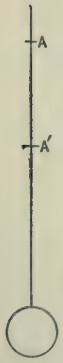

Fig. 43.

149. Corrections. Corrections are applied to observed results in pendulum measurements for:

1. Amplitude of arc of vibration.

Semi-arc used by Kater (seconds pendulum) = 1° 15′; by Mendenhall (half-seconds pendulum) = 30′. For seconds pendulum, amplitude 48′, correction is 1.05 seconds per day.

2. Temperatures of pendulum and of measuring rods. Lengths must be reduced to standard temperature.

Use of "dummy" pendulum for former. "Pendulum coefficient" determined experimentally by swinging at known temperatures.

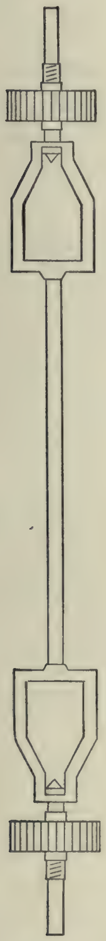

Fig. 44.

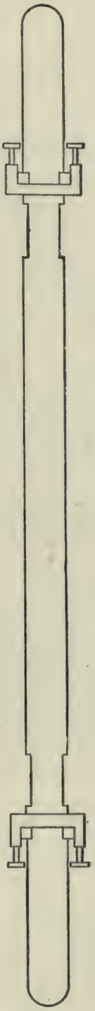

Fig. 45.

Fig. 46.

3. Reduction to vacuo: Corrections for buoyancy of air, air-drag, viscosity.

Bessel devised a reversible pendulum, symmetrical in shape about its middle point and with the knife-edges equidistant therefrom. One end of the pendulum was made heavier than the other, and the knife-edges were placed so as to be approximately at corresponding centers of oscillation and suspension. With such a form the air-correction disappears in the calculation. Fig. 44, shows Repsold's Bessel's pendulum, Fig. 45 shows Peirce's (U. S. C. S.), which is hollow, but loaded at one end.

In many cases it is preferable to swing the pendulum in an air-tight case under diminished pressure; e.g., 60 mm. at 0° C., as nearly as may be, and to reduce to this exactly, the pressure correction being determined experimentally for the individual pendulum.

4. Clock correction.
5. Flexibility of support of pendulum.
6. Reduction to sea level; as g diminishes with altitude.

There are also certain other minor corrections which are necessary to ensure the greatest accuracy.

150. Results of Measurement. The following are measured values of the length of the pendulum beating seconds and of g at certain places, according to data given in the Reports of the Coast and Geodetic Survey of the United States.

Station	Latitude	Elevation m.	Length cm.	g cm.
Washington, U. S. C. & G. S.	38° 53′ 13″	14	99.3047	980.098
Boston, State House	42° 21′ 33″	22	99.3335	980.382
Paris, Observatory	48° 50′ 14″	74	99.3863	980.903
Greenwich, Observatory	51° 28′ 40″	48	99.4134	981.171
Berlin, Observatory	52° 30′ 16″	35	99.4230	981.265

151. Uses of Pendulum.—1. *Determination of Value of g.* (Huygens, Paris, 1673.)

$$t=\pi\sqrt{\frac{l}{g}} \quad \therefore \quad g=\frac{\pi^2 l}{t^2}.$$

$\omega = \sqrt{\frac{g}{l}}$

$\therefore$

(*a*) Determine l_s at different stations. 1. By direct measurement of l_s (Huygens). 2. By comparison of period of invariable pendulum swung at different places; $g \propto \frac{1}{t^2}$. (Bouguer, Andes, 1737.)

Bouguer used a sphere suspended by a thread of constant length. Kater and Sabine (1820–25) employed an invariable seconds pendulum consisting of a bar terminated by a disk-shaped bob, the Indian

Trigonometrical Survey (1865–75), a Repsold's Bessel inversion pendulum (Fig. 44). Von Sterneck (1880), and Mendenhall, U. S. C. & G. S. (1890) used half-seconds pendulums. Fig. 46 shows that of Mendenhall. A quarter-seconds pendulum has also been used.

b. Study of diminution of g with elevation.

1. Determination of reduction factor to sea-level. (Bouguer, Andes, 1737.)

2. Determination of mean density of earth (Δ). (Carlini, Alps, 1821). Conversely, determination of mean density of mountain from Δ. (Mendenhall, Fujiyama, 1880.)

Mendenhall found density of Fuji $= 2.02$. Putnam (1894) found density of Pike's Peak to be $\delta = 2.57$; or $\Delta = 5.63$, assuming $\delta = 2.62$.

c. Study of change of g below surface of earth. (Airy, 1826, 1854.) Frequency of seconds pendulum at depth of 1250 ft. in Harton Coal Pit increased $2\frac{1}{4}$ oscillations in twenty-four hours, denoting increase of g of 1 part in 19,200.

From this it follows that g at surface is greater than if there were no shell of matter above base of pit by 1 part in 14,000; from which a value of Δ can be determined.

Von Sterneck (1883) from observations down to a depth of 1000 m. has determined a formula representing the variation of g with depth.

The pendulum methods of determining Δ are not comparable in accuracy with the methods of Cavendish and Poynting. The same is true of the determination of Δ from observations on the deflection of the plumb line or spirit level by a mountain—a method employed by Bouguer on Chimborazo, in 1740, and Maskelyne on Schehallien, 1774.

d. Study of local variations in g. Value is less than normal in mountainous regions and greater on islands and borders of continents. Observations made among the Himalayas by Indian Survey up to an elevation of 15,408 ft. (Basevi). In Europe by Defforges, in United States by Defforges and Putnam.

OBSERVATIONS OF PUTNAM

	Elevation m.	Observed value of g reduced to sea level cm.	Computed value of g cm.
Washington	14	980.101	980.087
Denver	1638	979.941	980.156
Pike's Peak	4293	979.844	980.083
Yellowstone Canyon	2386	980.369	980.605

e. g is independent of mass and material. (Newton, 1687; Bessel, 1832). Mass and material of bob of pendulum varied; time of vibration remains unchanged.

II. *Determination of Figure of Earth.* (Richer, 1672; Newton, 1687; Clairaut, 1743.)

From the law of variation of g with latitude it is possible to determine the ellipticity or oblateness of the earth. Ellipticity $\epsilon = \frac{a-b}{a}$.

Clairaut showed (1743) that for a spheroid of equilibrium of small oblateness composed of concentric strata each of the same density throughout, which, presumably, is approximately true of the earth as a whole, the ellipticity is $\epsilon = \frac{5}{2}\gamma - \zeta$ where γ is the ratio at the equator of the centrifugal force to gravity and ζ the total fractional increase of gravity from equator to pole. At a much later date (1849) Stokes showed that for such a spheroid no particular law of density need be assumed.

Clairaut also showed that the value of g in any latitude λ is given by the formula $g_\lambda = g_0(1+\zeta \sin^2 \lambda)$. It is easily seen that in the formula $\zeta = \frac{g_\lambda - g_0}{g_0 \sin^2 \lambda} = \frac{g_{90} - g_0}{g_0}$.

It follows from these facts that the value of the polar flattening can be determined from a comparison of pendulum measurements in different latitudes.

Approximately calling $\gamma = \frac{1}{288.4}$ and $\zeta = \frac{1}{188.6}$ (its value as found experimentally by the pendulum), $\epsilon = \frac{1}{297.1}$. The calculations leading to exact results are, of course, very complex. The most probable value of ϵ as found by the pendulum method is $\frac{1}{298.3}$ (Helmert, 1901, 1907). The geodetic method gives $\frac{1}{293.5}$ (Clarke, 1878), but this method is considered to be less trustworthy than the pendulum method owing to the comparatively small portion of the earth's surface which has been accurately triangulated. A very recent determination (1906) by the U. S. C. & G. S., using the geodetic method and based on measurements in this country gives $\epsilon = \frac{1}{297.8}$. From a consideration of results of all the various methods which have been employed the value $\frac{1}{298.3}$ has recently been reached by Helmert (1907) as the most probable one, agreeing with the results from pendulum measurements alone.

The difference between the greatest and the least value of g is about $\frac{1}{189}$ of the minimum value, as stated above.

The actual solid dealt with in all these calculations is the *geoid,* i.e., the spheroid whose surface would everywhere coincide with the ocean level.

Formulæ. The normal value of g for any latitude λ, at sea-level, may be computed from the following formula given by Putnam (1897):

$$g_\lambda = 978.066\,(1 + 0.005243 \sin^2 \lambda).$$

The following particular values are thus obtained:

Place	Latitude	Value of g cm.
	0°	978.066
	45°	980.630
	90°	983.194
Washington, U. S. C. & G. S......	38° 53′ 13″	980.087
Boston, State House.............	44° 21′ 53″	980.394
Paris, Observatory...............	48° 50′ 14″	980.972
Greenwich, Observatory..........	51° 28′ 40″	981.205
Berlin, Observatory..............	52° 30′ 16″	981.294

Values of g determined by actual measurement at the five particular stations given above are stated in §150.

These results should be reduced to sea-level for purposes of exact comparison with the values calculated from the general formula.

A later and, perhaps, somewhat more precise formula than the above is given by Helmert (1901); viz.

$$g_\lambda = 978.046\,(1 + 0.005302 \sin^2 \lambda - 0.000007 \sin^2 2\lambda).$$

The difference, however, in the results given by the two formulæ is very slight.

For an elevation h we may use the formula $g_h = g\left(1 - \frac{2h}{R}\right)$, or such other as may seem preferable in any particular case.

A formula which has been widely used for the reduction to sea-level is that of Bouguer, in which for the term of correction $\frac{2h}{R}$ is substituted $\frac{2h}{R}\left(1 - \frac{3\delta}{4\Delta}\right)$. Δ is the mean density of the earth, and δ the density of the adjacent matter lying above sea-level. Strictly considered, Bouguer's correction applies to an extended plateau. The added term is introduced in order to take account of the attraction of the mass of elevated land on which the station is situated.

VALUE OF g IN DIFFERENT LATITUDES

(From Everett's "C. G. S. System.")

	Lat.	g (measured) cm.	Elevation m.
Spitzbergen	79° 50′	983.08	6
St. Petersburg	59° 56′	981.90	8
Berlin	52° 31′	981.27	35
London	51° 31′	981.18	28
Dunkirk	51° 2′	981.14	0
Paris	48° 50′	980.92	70
Padua	45° 24′	980.67	31
Fiume	45° 19′	980.63	65
Bordeaux	44° 50′	980.54	17
Boston	42° 22′	980.38	22
Washington	38° 53′	980.10	10
Cape Town	33° 56′S	979.64	10
Port Jackson	33° 52′S	979.67	
Calcutta	22° 33′	978.78	6
Bombay	18° 54′	978.61	11
Madras	13° 4′	978.20	8
Sierra Leone	8° 29′	978.18	58
Para	1° 27′S	978.03	12

Oceanic Islands

Ascension	7° 56′S	978.29	5
St. Helena	15° 56′S	978.65	9
Mauritius	20° 10′S	978.87	

The values of g for London, Boston and Washington have been added to Everett's table from other sources.

The following values of g were obtained by Putnam in connection with the M. I. T. party on the Peary Expedition to Greenland in 1896. A Mendenhall half-seconds pendulum was used.

	Latitude	g cm.	Elevation m.
Washington, U. S. C. & G. S.	38° 53′ 13″	980.098	14
Sydney, C. B.	46° 08′ 32″	980.720	11
Ashe Inlet, Hudson Strait	60° 32′ 48″	982.105	15
Umanak, Greenland	70° 40′ 29″	982.590	10
Niantilik, Cumberland Sound	64° 53′ 30′	980.273	7

III. *Regulation of Clocks.*

The application of the pendulum to the regulation of clocks is generally accredited to Huygens (1656), although he appears to have been, in fact, anticipated by Galileo. Rival claims are advanced for Hooke, Harris and others.

A circular pendulum with a spring suspension is universally used for this purpose.

Escapement: Crown-wheel (De Vick, 1360, adopted for pendulum by Huygens, 1656); Anchor (Hooke, 1656); Dead Beat (Graham, 1715); Gravity (Mudge, 1760); Free escapement, impulse communicated through suspension springs (Riefler, 1889).

Compensation pendulums: Graham, mercurial (1721); Harrison, gridiron, brass and iron rods (1726); Reid, modification using zinc and steel rods (1812); Riefler, rod of *invar* (Guillaume's nickel-steel alloy) with short compensation tube of steel and brass (1898).

A standard clock should be kept in a constant temperature room.

Compensation may be secured for barometric changes by the action of a magnet governed by a barometer upon a second magnet carried by the pendulum, as in the standard Greenwich Observatory clock. Or, better, the clock may be kept in an air-tight case under reduced pressure (e.g., 675 mm.), and so protected from atmospheric variations. This was done as early as 1867 in the Berlin Observatory, and is now common. Such a clock is automatically wound by electricity.

A standard astronomical clock by Riefler at the U. S. Naval Observatory, Washington, placed in an air-tight case has run with a mean daily variation of only 0.015 second for a period of $3\frac{1}{2}$ months. This clock possessed a nickel-steel pendulum rod and was kept in a constant temperature room.

IV. *Former Standard of British Weights and Measures.*

As the result of measurements in 1817 and subsequent years Kater had determined the length of the seconds pendulum at London reduced to vacuum and sea-level to be 39.1393 in. The British Weights Measures Act of 1824 constituted as the legal standard of length for Great Britain the brass yard made by Bird in 1760, a line standard, in terms of which Kater's pendulum had been measured. It furthermore provided that in case of loss or destruction of the standard yard it should be reproduced by constructing a new standard bearing the same proportion to the length of the seconds pendulum as that borne to it by the original standard. In 1834 the Imperial standards of weights and measures, including the yard of 1760, were destroyed in consequence of the burning of the Parliament Houses. Meanwhile various sources of error had been discovered in Kater's pendulum measurements, and it became evident that a much closer approximation of the lost standard yard could be made from a comparison of the best existing copies than by employing the method of restoration set forth in the Act of 1824. A new line-standard bar of bronze (Baily's metal) was, therefore, constructed by the method of copying. This was constituted the legal Imperial standard yard by the Weights and Measures Act of 1855.

Under the Act of 1824 the pound (troy of 5760 grains) was defined by reference to the weight of a cubic inch of water at 62° F. as determined by Shuckburgh.

The Imperial pound of 1855, an avoirdupois pound of 7000 grains is represented by a cylindrical mass of platinum, and is in no way referred to any other quantity.

The length of the seconds pendulum, therefore, formed the legal basis of the British system of weights and measures only from 1824 to 1855.

V. *Physical Demonstration of the Rotation of the Earth.* (Foucault, 1851.)

If a pendulum could be suspended at the north pole directly in the line of the earth's axis, and set into vibration in any chosen plane, a line marking the horizontal trace of the plane of vibration in its original position would rotate about the earth in a left-handed direction at the rate of 15° per hour. Because of the permanence of the plane of vibration of the pendulum this would remain unchanged in position. Hence, referred to the surface of the rotating earth the plane of vibration would appear to rotate right-handedly at the same rate.

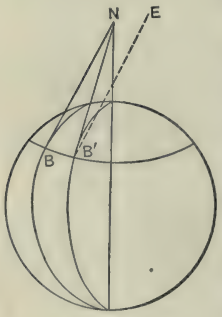

Fig. 47.

In a lower latitude, as e.g., at B (Fig. 47), the pendulum in the actual experiment is set into vibration in the meridian BN. Somewhat later B will have moved to B'. But the horizontal trace of the vertical plane in which the pendulum continues to move remains parallel to its original direction, and is $B'E$, which is no longer north and south in direction, but inclined to the meridian by an angle $NB'E$ dependent on the latitude. The hourly apparent rotation of the plane of vibration for latitude λ is $\theta = 15° \sin \lambda$.

For $\lambda = 45°$, $\theta = 10° 36'$.

At the equator $\theta = 0$.

At the poles $\theta = 15°$.

A gyroscope suspended so as to move freely about a vertical axis may be used for the same purpose. The wheel with its axle horizontal is caused to rotate with great rapidity. If the axle is placed initially in the meridian, the revolution of the earth causes an apparent displacement of the plane of rotation of the wheel which is of the same character as that of the plane of vibration of the pendulum previously described.

In Foucault's original experiment, performed in the Pantheon, Paris, the pendulum was 220 ft. in length. The motion was shown by causing a pointed spindle projecting below the cannon ball which formed the bob to swing through an arc of moist sand.

Permanent records were obtained in 1876 by the use of a smoked glass plate on which the trace of the plane of vibration was registered by means of a style.

NOTES ON GRAPHICAL STATICS

1. Graphical Methods of determining the resultant of a system of coplanar forces acting on a framed structure and the stress in the members.

Problem 1. *To determine the intensity and direction of the resultant in the case of concurrent coplanar forces.*

In Fig. 1 let AB, BC, and CD be the action lines of three forces forming a concurrent system whose resultant is desired. Let the intensity of each force be represented by the length of its action line. Let the unit of length of the scale represent an intensity of 1 lb.

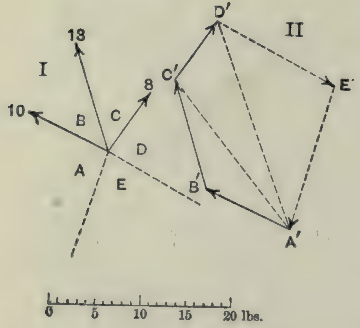

FIG. 1.

From any point, as A', draw a line $A'B'$ parallel to force AB and make it equal to ten linear units; from B' draw a line $B'C'$ parallel to force BC and make it equal to thirteen linear units; from C' draw the line $C'D'$ parallel to force CD and make it equal to eight linear units; complete the polygon by drawing the line $A'D'$. If through the common vertex we draw a line parallel to $A'D'$, it will be the action line of the resultant of AB, BC, and CD, and its intensity will be the number of linear units in the length $A'D'$. This results from the principle of the parallelogram of forces, since by construction $A'C'$ must be equal and parallel to the resultant of AB and BC, and $A'D'$ must be equal and parallel to the resultant of $A'C'$ and CD, or of the three forces given.

2. Graphic Nomenclature.—In Fig. 1 the diagram $ABCD$ shows the actual positions as well as the directions of the known forces in the plane of construction; it is, therefore, called a *position diagram.* In this diagram the forces are indicated by the letters written between the action lines.

The diagram $A'B'C'D'$ is called the *force polygon,* and always gives the intensities of the forces it represents. It will also give the directions of the forces, if care is taken to draw all the known forces in the directions in which they act; thus the force AB acts in the direction $A'B'$. Each force in the force polygon is indicated by the same letters as in the position diagram, but in the force polygon the letters are written at the ends of the forces and are distinguished by accents. In constructing the force polygon $A'B'C'D'$ the forces were drawn in the direction of the arrow-heads, and $A'D'$ was found to be the direction and intensity of the resultant. As a similar result would have been obtained in solving any other system of concurrent forces, we may conclude that:

If the forces in the force polygon are drawn in the direction in which they act, the closing line, drawn from the origin to the extremity of the last force, will represent the intensity and direction of the resultant of the forces.

3. Problem 2. *To find the unknown intensities of two forces of a concurrent system* **in equilibrium** *when the action lines of all the forces, and the intensities of all the other forces, are known.*

In Fig. 1 let AB, BC, and CD be the action lines of the known forces, and DE and EA be the action lines of the unknown forces whose intensities are desired. As before, construct the force polygon $A'B'C'D'$; from D' draw the line $D'E'$ parallel to DE, and from A' the line $A'E'$ parallel to AE. Then the number of linear units in $D'E'$ and $E'A'$ will be the intensities in pounds of the unknown forces. According to Problem 1, $A'D'$ is equal and parallel to the resultant of AB, BC, and CD, and $D'A'$ is equal and parallel to the resultant of DE and EA. Since $A'D'$ and $D'A'$ are equal and contrary, the five forces AB, BC, CD, DE, and EA must form a concurrent system in equilibrium, and $D'E'$ and $E'A'$ must be the intensities of the unknown forces.

4. Other Uses of the Force Polygon.—The force polygon may be utilized to determine whether a system of concurrent forces is *in equilibrium* or has a resultant. If the force polygon closes, the sum of the horizontal and the sum of the vertical components of the system of forces must each be equal to zero, and the concurrent system must be *in equilibrium.* Conversely, if the concurrent system of forces is *in equilibrium,* its force polygon must close.

Any force of the system may be resolved into two components acting along any desired lines of direction. Thus the force $D'A'$, Fig. 1, may be resolved into two components parallel to DE and AE, by simply drawing through the extremities of $D'A'$ the lines $D'E'$ and $A'E'$, parallel respectively to DE and AE.

The force polygon may be utilized *in determining two unknown* *directions in a concurrent system of forces when all the intensities are known.* If the intensities instead of the directions of AE and ED were given, by describing arcs with A' and D' as centers and the intensities as radii the point E' could be determined. In the general case the arcs will intersect in two points and give two solutions of the problem.

The force polygon may also be utilized *in determining one unknown intensity and one unknown action line.* Thus, if the action line of DE and the intensity of AE were given, the point E' could be determined by drawing $D'E'$ parallel to DE, and from A' as a center with the intensity of EA as a radius describing an arc intersecting $D'E'$. Under these conditions there would also be two solutions in the general case. If both the intensity and the action line of either DE or EA were given, there would be but one solution.

5. Non-concurrent Forces—Problem 3.—*To determine the resultant of a system of non-concurrent coplanar forces.*

In Fig. 2 let AB, BC, and CD be the action lines of the three forces whose intensities are given by the force polygon $A'B'C'D'$; required the resultant.

The direction and intensity of the resultant is given by the closing line of the force polygon $A'D'$. This results as before from the principle of the parallelogram of forces, since by construction $A'C'$ must be equal and parallel to the resultant of AB and BC, and $A'D'$ must be equal and parallel to the resultant of $A'C'$ and CD.

The resultant will be fully known, therefore, whenever we know the position of one point of its action line.

When the action lines of the forces intersect within the limits of the drawing we may employ the following method based on Problem 1. Produce the action lines of AB and BC until they intersect. Through the point of intersection draw their resultant, which will be parallel to $A'C'$. Produce this resultant until it intersects the action line of CD; their point of intersection will be one point on the action line of the resultant, and the resultant itself will be AD, drawn through this point parallel to $A'D'$, as shown in Fig. 2, diagram I.

When the action lines of the forces do not intersect within the limits of the drawing we must employ the following method:

Assume a point O, Diagram II, called the *pole* of the force polygon, at any convenient point in the construction plane, and draw the lines $A'O$, $B'O$, $C'O$, and $D'O$. These lines may be taken as the intensities and directions of a new system of forces which are components of the original system. $A'O$ and OB' will be the intensities and directions of the components of AB, and $B'O$ and OC' will be the intensities and directions of the components of BC, etc.

If we assume any point on the action line of the force AB, as o', and draw the lines $o'a$ and $o'b$ equal and parallel to $A'O$ and OB', the original force AB may be replaced by its components $o'a$ and $o'b$. If we find the point o'' where the line bo' intersects the action line of the force BC, we may replace the force BC by its components $o''b$ and $o''c$, equal and parallel to $B'O$ and OC'. If we find the point o''' where the line $o''c$ intersects the action line of the force CD, we may also replace the force CD by its components $o'''c$ and $o'''d$, equal and parallel to $C'O$ and OD'.

The resultant of these six components, $o'a$, $o'b$, $o''b$, $o''c$, $o'''c$, and $o'''d$ must also be the resultant of the original forces AB, BC, and CD.

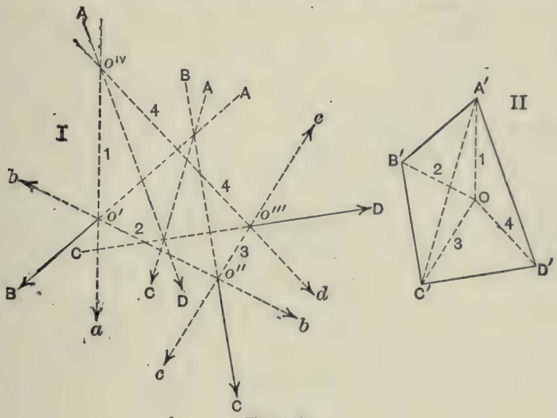

FIG. 2.

The components $o'b$ and $o''b$ are equal and directly opposed; hence their resultant is zero. For like reason the resultant of $o''c$ and $o'''c$ is zero; hence the resultant of $o'a$ and $o'''d$ must be the resultant of the system. The resultant of the system, AD, therefore, acts through the point o^{IV}, the point of intersection of $o'a$ and $o'''d$, and is parallel to $A'D'$. This is evident from the force polygon, since $A'O$ and OD' are the components of $A'D'$ in that polygon. The lines $o'a$ and $o'''d$ are in this case called the *closing lines* of the polygon $o'o''o'''o^{\mathrm{IV}}$.

The polygon $o'o''o'''o^{\mathrm{IV}}$ is called the *equilibrium polygon* or *polar polygon*, its sides being respectively parallel to the polar lines from the pole o.

From the construction it follows that the point of the resultant of any system of non-concurrent forces not in equilibrium may be determined by finding the point of intersection of the two closing lines of the equilibrium polygon.

6. Problem 4.—*To determine the intensities of two unknown parallel forces in a system of non-concurrent forces in equilibrium when only the points of application of the unknown forces are given.*

In Fig. 3 let the forces AB, BC, and CD, whose intensities are given by the force polygon $A'B'C'D'$, form a non-concurrent system of forces in equilibrium with two unknown parallel forces which act through the points a and b. If the unknown forces are parallel, they must also be parallel to the resultant of the forces AB, BC, and CD in order that the sum of the horizontal and the sum of the vertical components of the system shall each be equal to zero.

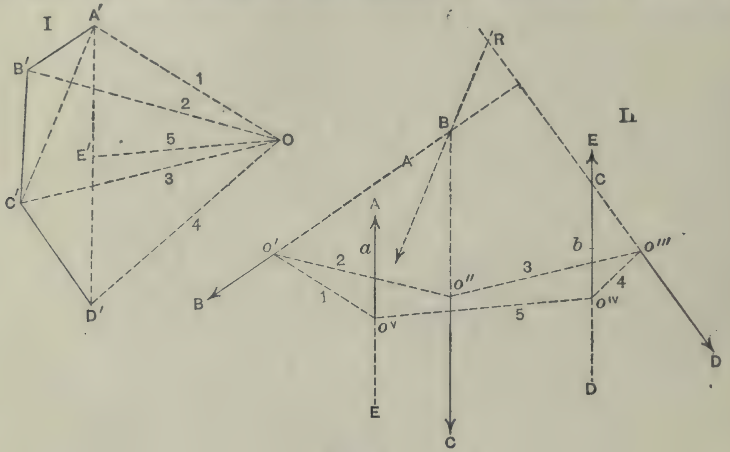

Fig. 3.

In the force polygon draw the line $A'D'$; it will be equal and parallel to the resultant of the forces AB, BC, and CD, and must be equal to the sum of the intensities of the two unknown forces. By hypothesis the lines DE and EA drawn through the points a and b parallel to $A'D'$ must be the action lines of the unknown forces.

When the position of the resultant of the known forces can be determined without constructing an equilibrium polygon we may employ the following method to determine the intensities of the unknown forces:

Let AB (Fig. 4) be the action line of the resultant of the known forces, and BC and AC be the action lines of the two unknown parallel forces. Let $A'B'$ be the intensity of the force AB. From any point on AB, as e, draw def perpendicular to AB. Then will ef

and de be the lever-arms of the forces AC and BC with respect to any point on AB. Since for equilibrium the resultant of BC and CA must be equal and opposed to AB, we have, from the principle of parallel forces,

$$\text{Intensity } BC \times ed = \text{intensity } CA \times ef. \quad . \quad . \quad . \quad . \quad (1)$$

To determine the unknown intensities of BC and CA, from B' draw $f'd'$, making any convenient angle with $A'B'$. Lay off $f'e' = fe$, and $e'd' = ed$, and draw $d'A'$ and parallel to it $e'C'$. Then

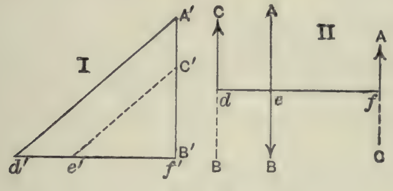

FIG. 4.

will $B'C'$ be the intensity of the force BC, and $C'A'$ the intensity of the force CA; since from the similar triangles we have

$$B'C' : C'A' = e'f' : e'd' :: ef : ed,$$

or

$$B'C' \times ed :: A'C' \times ef. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When the position of the resultant of the known forces can be determined only by constructing an equilibrium polygon we must employ the following method:

In Fig. 3 assume O as the pole and draw the lines OA', OB' OC', and OD'. Then from any point on the action line of the force AB, as o', construct the equilibrium polygon $o'o''o'''o^{\text{IV}}o^{\text{V}}$.

Since the action line of the force EA, acting through a, intersects the line $o'o^{\text{V}}$ at o^{V}, we may lay off from the point o^{V}, on the line $o^{\text{V}}o'$, an intensity equal to OA' and consider this as one of the components of the unknown force EA. In the same manner we may lay off from o^{IV} on $o^{\text{IV}}o'''$ an intensity equal to $D'O$, and consider it one of the components of the unknown force DE acting through b.

In the concurrent system of forces acting at o^{V} one of the angles of the equilibrium polygon $o'o''o'''o^{\text{IV}}o$, we have the component of EA acting along $o^{\text{V}}o'$ fully given and the action line of the component acting along $o^{\text{V}}o^{\text{IV}}$, and the action line of the force itself, EA; hence we may determine the intensities of the unknown component and the force EA by drawing lines through the points O and A' of the force polygon, parallel to $o^{V}o^{IV}$ and EA. In a similar manner we may determine the intensity of the force DE and of its component acting at o^{IV} along $o^{IV}o^{V}$ by drawing lines in the force polygon through O and D' parallel to $o^{IV}o^{V}$ and ED. We thus determine the point E' and the intensities of the two unknown forces; $E'A'$ is the intensity of the force acting through a, and $D'E'$ is the intensity of the force acting through b. In this case the line $o^{IV}o^{V}$ is the closing line of the equilibrium polygon.

Hence we see that to find the intensities of two unknown parallel forces of a non-concurrent system in equilibrium we need only draw, in the force polygon, a line through the pole parallel to the closing line of the equilibrium polygon. It will divide the resultant into two parts, each of which represents the intensity of one of the unknown forces.

7. Problem 5.—*To determine the intensities of two unknown forces in a system of non-concurrent forces in equilibrium when only the points of application of both unknown forces and the action line of one of them are given.*

Let the full lines AB, BC, and CD (Fig. 5) be the action lines of the known forces whose intensities are given by the force polygon $A'B'C'D'$; let the full line DE be the action line of one of the unknown forces, and let o' be a point on the action line of the other unknown force.

I. When the action line of the resultant of the known forces can be determined without the aid of the equilibrium polygon we may use the following methods:

First. When the action line of the resultant and that of one of the unknown forces intersect within the limits of the drawing. In this case the action line of the other force may be determined by connecting its given point of application with the point of intersection. Thus, in Fig. 5, we construct AD parallel to $A'D'$, the resultant of the known forces, and find its point of intersection with DE; the action line of AE will pass through o' and this point of intersection. This construction results from the fact that three forces in equilibrium must form either a parallel or a concurrent system of forces. Having given the action lines of the unknown forces, their intensities may be found from the force polygon.

Second. When the action line of the resultant and that of one of the unknown forces do not intersect on the drawing. Since the moment of the resultant with respect to o' must be equal to moment of *DE* with respect to the same point, we may employ a method similar to that shown in Fig. 4.

In diagram IV, Fig. 5, let o' represent the point o' of diagram II, *AD* represent the action line of the resultant *AD*, and *DE* the action line of the unknown force *DE*; then will $o'a$ be the lever-arm of the resultant, and $o'b$ be the lever-arm of the force *DE* with respect to the point o'. In diagram III, Fig. 5, make $A'D'$ equal $A'D'$ in diagram I, and make $o'b$ and $o'a$ equal to $o'b$ and $o'a$ in diagram IV. Connect o' with D' and draw parallel to it the line aE'; then $D'E'$ will be the intensity of the force *DE*. Since from the similar triangles $o'A'D'$ and $aA'E'$ we have

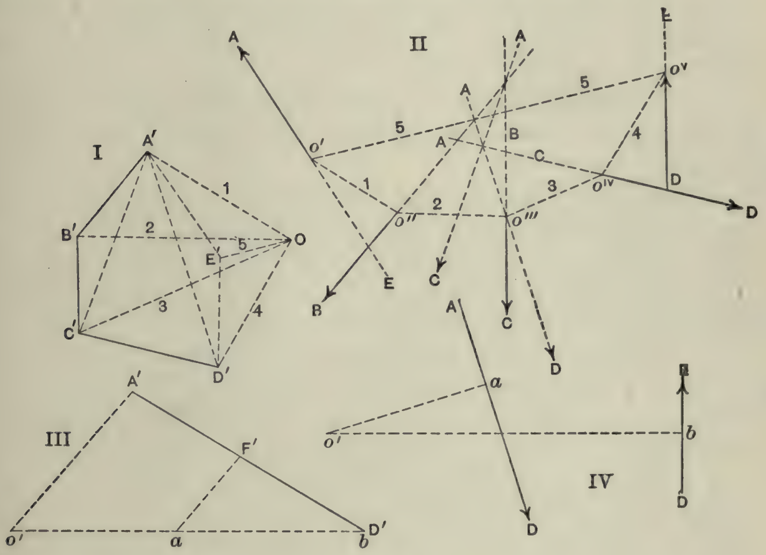

FIG. 5.

$$D'A' : D'E' :: o'b : o'a,$$

hence

$$D'A' \times o'a = D'E' \times o'b. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Having determined the intensity *DE* in I, lay off $D'E'$ parallel to *DE* and equal to $D'E'$(III); $A'E'$ will be parallel to the action line of *EA* and will represent its intensity.

II. When the action line of the resultant cannot be determined without the use of the equilibrium polygon we must employ the following method:

Assume the pole O and draw the lines OA', OB', OC', and OD'; then from the preceding problem OA' may be taken as one of the components of the force EA whose direction and intensity are required, and $D'O$ as one of the components of the force DE. From o', the known point on the action line of the force EA, as an origin, construct the equilibrium polygon $o'o''o'''o^{\text{IV}}o^{\text{V}}$. The closing line $o^{\text{V}}o'$ will be the action line of the other component of the forces DE and EA. Hence if in the force polygon we draw OE' parallel to $o^{\text{V}}o'$, and $D'E'$ parallel to DE, $E'A'$ will be the intensity and direction of the force EA, and $D'E'$ will be the intensity of the force DE. To complete the solution the line EA should be drawn through o' parallel to $E'A'$.

It will be observed that when only one point on the action line of an unknown force is given, the equilibrium polygon must begin at that point.

8. Uses of the Equilibrium Polygon.—In a system of non-concurrent forces, to determine whether the forces are in equilibrium or have a resultant, we must utilize both the force and equilibrium polygons. If the force polygon closes, the sum of the horizontal and the sum of the vertical components of the system must each be equal to zero. If the equilibrium polygon closes, the sum of the moments of the forces with respect to any point in the plane of the forces must be equal to zero. Hence, if the force and equilibrium polygons of a non-concurrent system of forces both close, the system is in equilibrium, and conversely the force and equilibrium polygons of a non-concurrent system of forces in equilibrium must close.

If the resultant of a system of forces is a single couple, the force polygon will close, but the equilibrium polygon will not. Hence, we may assume that a non-concurrent system whose force polygon closes but whose equilibrium polygon does not may be reduced to a single couple.

9. Solution by Reciprocity of Figures.—It will be noted that the Polar or Equilibrium Polygon and Force Polygons are reciprocal figures. *That is, the lines forming a triangle in one polygon are parallel to the lines from a vertex in the other.*

This principle may be applied in solving the problem of three unknown forces of a coplanar system in equilibrium. In this case the solution simply depends on producing two reciprocal polygons, as outlined by the following example.

10. The problem of finding three non-concurrent unknown forces of a coplanar system in equilibrium, Fig. 6, may be solved by constructing a force polygon for the system. To explain how this may be done, assume the forces known, as in Fig. 6; draw a force polygon, assume a pole, and draw a polar polygon, Fig. 7. The system is in equilibrium, and both the force and polar polygons close.

Add the line *Pe* to the force polygon diagram, also the lines $1''h$ and *hB* to the force diagram. Then it will be seen that the figures *PecdP* and $h1''B2''h$ are reciprocal. (See definition, par. 9.)

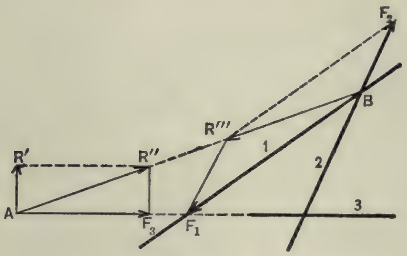

Fig. 6.

Considering the forces 1, 2, 3 unknown, we may draw the definite line *ab* of the force polygon, which will in general be the closing line of a force polygon for the known forces of the system. We may also draw the indefinite sides from *a* and *b* parallel to 3 and 1, the polar lines *Pa* and *Pb*, and the sides $3''R''_k$ and R''_k1'' of the polar polygon. The problem now consists in locating the side *cd* of the force polygon by the reciprocity of *PecdP* and $h1''B2''h$.

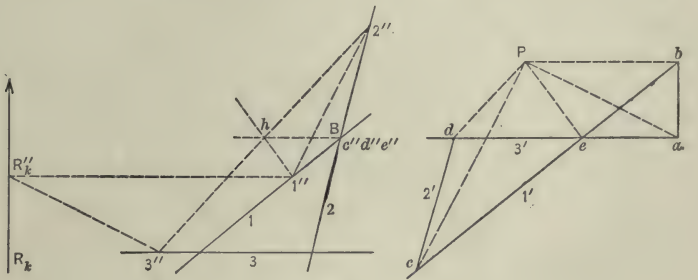

Fig. 7.

The vertices $1''$ and *B* are already established, and it is known that *ecd* (*cd* unknown) is a triangle. Then, according to the distinctive property of reciprocal figures, the lines radiating from *B* must be parallel to the sides of the triangle *ecd*. Two are already so drawn, 1 parallel to $1'$ and 2 to $2'$; hence *Bh* may be drawn parallel to $3'$. That is, through the intersection of the two action lines parallel to the two sides of the force polygon following *ab*, draw a line parallel to the other action line.

Next draw *Pe*, and establish *h* by a parallel to *Pe* through $1''$, since we know that lines parallel to *ec*, *ed* and *eP* must form a triangle in the reciprocal $h1''B2''h$. Now join *h* and $3''$, and draw *Pd* parallel to $h3''$, thus establishing *dc* and completing the force polygon.

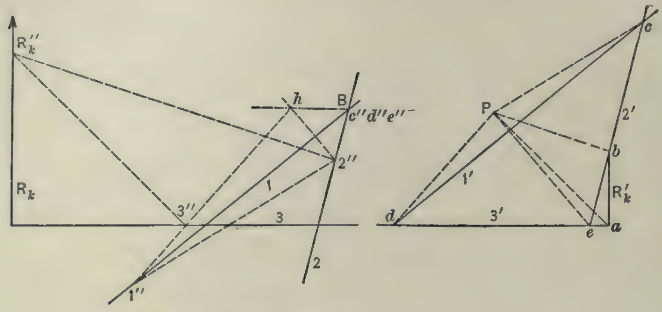

FIG. 8.

Check the accuracy of the diagram by completing the reciprocal figures.

11. Fig. 8 shows the solution of the same problem by establishing the side $1'$, closing on $2'$ and $3'$.

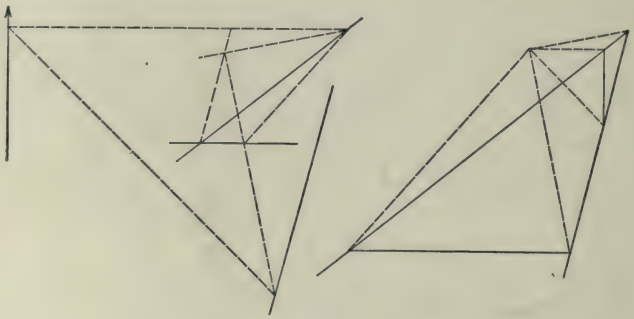

FIG. 9.

Draw $3''R''_k2''$, then *Bh* parallel to 3, establish *h* by the intersection of *Bh* and $2''h$ ($2''h$ parallel to *Pe*) draw $h3''$ establishing $1''$, draw $1''2''$, and then complete the reciprocity of *PcedP* to $hB2''1''h$.

Ex. 1.—Solve the same problem by closing $3'$ on $1'$ and $2'$ (Fig. 9). (Letter the figure.)

Ex. 2.—Find the resultant of a couple by the force and polar polygons.

APPLICATION TO FRAMED STRUCTURES

12. In applying these principles to frame structures we assume the forces to act at the vertices of the frames, and that the stresses on the members are *wholly longitudinal.* If a load be given at any point other than a vertex, *then its parallel components at the adjacent vertices must be found and combined with the forces applied directly to these vertices before proceeding with the solution.* Wind pressure and weights of the parts of the structure are forces of this character, and in the general case must be taken into consideration. These forces are not given in the following problems, the solutions being limited to the stresses due to the given applied loads.

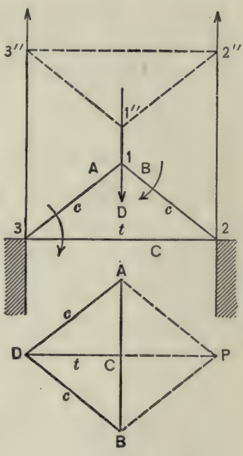

FIG. 10.

Beginning with a simple triangular frame supported at two vertices and loaded at the other, we have the solution given in Fig. 10.

The given system, including the reactions, is in equilibrium, and hence the force polygon must close.

The system of lettering is as follows: In the frame diagram the space between two adjacent action lines, running around the frame, and also each triangular space in the frame, is indicated by a letter; and any line in the figure is indicated by naming the letters which designate the spaces on opposite sides of the line. Thus, the given force is designated by AB, as in the position diagram, the right-hand reaction by BC, the left-hand reaction by CA, the horizontal chord of the frame by CD or DC, and the inclined sides by AD and BD. A vertex of the frame is named by the letters of spaces opposite, as *vertex* AC for the vertex at the left-hand support.

In the **stress diagram** (built up on the force polygon) an applied force or a stress is indicated by the letters which designate the parallel line in the frame diagram, but the letters are placed at the ends of the line. Thus, AB in the force polygon represents the intensity and the direction of the force whose action line is AB in the frame diagram.

Having marked the frame diagram in this manner, and also the first side of the force polygon, the first step is to establish the point C so that BC and CA shall complete the force polygon and give the reactions whose action lines are BC and CA in the frame diagram. To do this assume a pole, as P, and draw the polar lines PA and PB. Then draw the polar polygon $1''2''3''$, and a polar line PC, parallel to the last line of the polar polygon; the vertex C of the force polygon is thus determined. This is seen from the fact that both the force and polar polygons close; and hence having completed the polar polygon, the last polar line must be parallel to the side which closes the polar polygon.

Having completed the force polygon in this manner, the stress diagram is completed by drawing a force polygon for each vertex of the frame.

Begin at the vertex of the left-hand reaction, and mark the parts whose stresses are to be found, embracing them with an arc running right-handed from the line of the known reaction. Then starting with CA (known) in the force polygon, draw through A a parallel to AD of the frame. The closing line of the force triangle for vertex AC must close at C and be parallel to DC of the frame. Hence, draw through C a parallel to DC, establishing the point D and completing $CADC$.

The force AD in this force polygon acts along the beam AD TOWARD the vertex AC, and hence the stress in the beam AD is a *compression.* The force DC in $CADC$ acts along the chord DC AWAY FROM the vertex AC, and hence the stress in DC is a *tension.* From this discussion it is evident that the forces at any vertex act in the directions in which the corresponding force polygon is run, and the way to run the force polygon is always indicated by any known force on which the force polygon is constructed. Therefore the running of a force polygon always indicates whether a force acts toward or from the corresponding vertex, and hence whether the stress be compression or tension.

As soon as the character of the stress on a member of the frame has been determined, indicate it on the diagram by *c* for compression and *t* for tension.

In running a force polygon always begin with the first force known, ending with the line which closed the polygon.

To complete the stress diagram in Fig. 10, take the next vertex, AB. The known part of the force polygon for this vertex is DAB and the closing line BD gives the stress on BD of the frame, acting toward the vertex and therefore a compression.

The accuracy of the diagram is checked by the parallelism of stress BD to beam BD.

The polygon for vertex BC is already constructed, and may be run from any vertex, since all of the sides are known. It should be run to check the character of the stresses in CD and DB.

In a symmetrically loaded frame, as in Fig. 10, the point C in the stress diagram is midway between A and B, since the load is borne equally by the two supports. It is then unnecessary to make the construction for establishing C. It is given in this problem to explain the principle involved.

13. Directions for Solving Problems in Graphical Statics. In the exercises given in this subject, the dimensions of the figures are intended to be such that they may be reproduced on the cross-section paper so that parallelism may be accurately and quickly established. The vertices of trusses are taken so that they may fall on the intersections on the paper, thus greatly facilitating the construction of the diagrams.

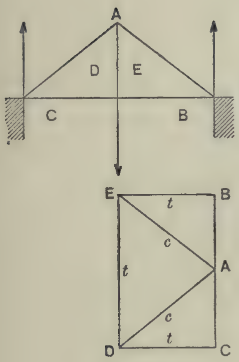

FIG. 11.

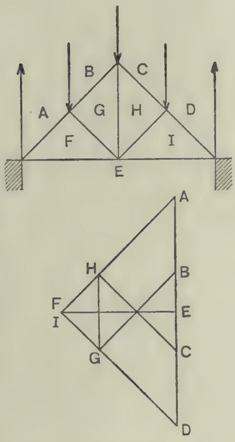

FIG. 12.

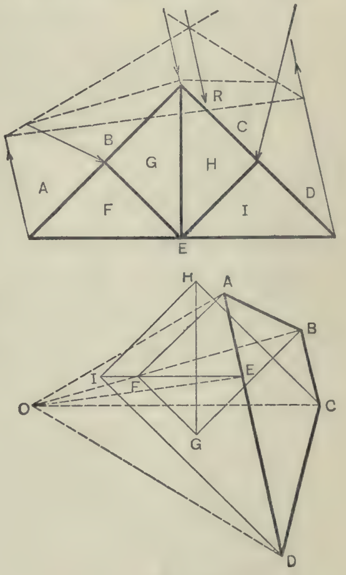

FIG. 13.

Ex. 3. *The King-post Truss.* The king-post truss consists of two inclined struts, a horizontal chord, and a vertical tie, as shown in Fig. 11. The diagram shows the solution for a load applied at vertex DB. *It us understood that DC and EB are separate beams jointed at the vertex DB, so that the whole load is first taken on the tie and then transmitted through the struts to the lower chord.* If DC and EB should constitute a continuous chord having transverse strength, then part of the given load would be taken on this chord.

Ex. 4. Fig. 12 shows the addition of two struts to the king-post truss, with equal loads at the upper vertices.

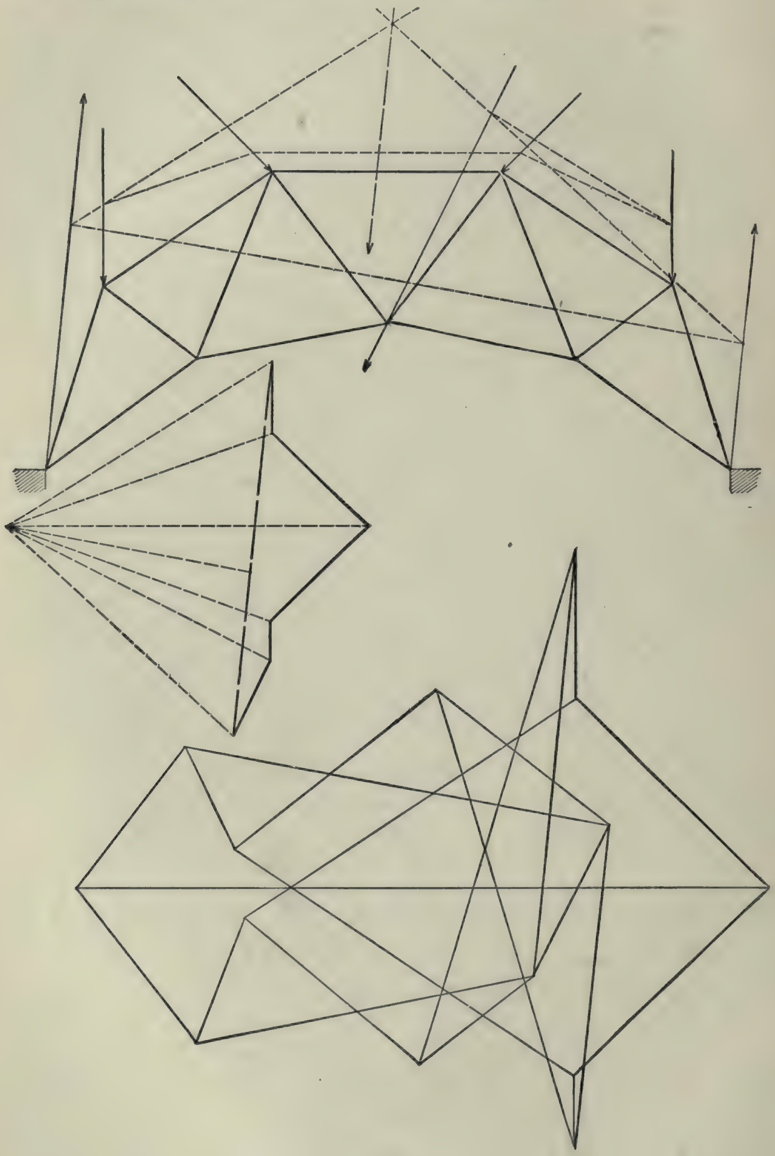

FIG. 14.

Ex. 5. Fig. 13 shows the same truss as Fig. 12, but with unequal loads.

Ex. 6. Fig. 14 shows a solution for an irregularly loaded truss. The scale of the stress diagram is double that of the polar figure by which the reactions are determined.

Letter the figures consistently according to the adopted notation.

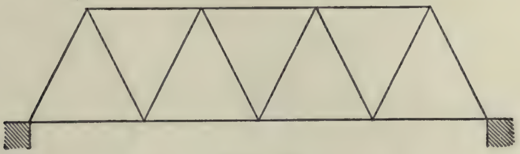

Fig. 15.

Ex. 7. Fig. 15 shows a simple Warren truss. Make the solution for a load uniformly distributed along the entire lower chord of the truss. Also for various given loads at the vertices.

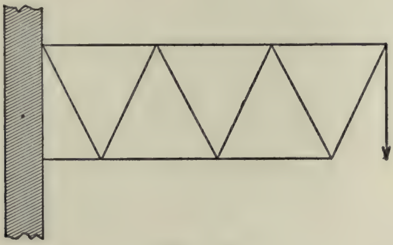

Fig. 16.

Ex. 8. Fig. 16 shows a cantilever in the form of a Warren truss. Make the solution for the given loads. Also find the reactions by the parallelogram of forces.

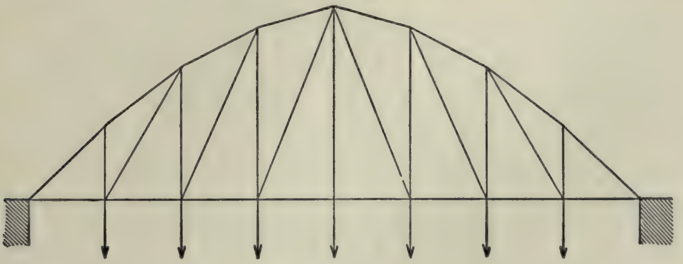

Fig. 17.

Ex. 9. Solve the problem of Fig. 17. (Note that the middle vertical member takes the middle load *directly*.)

Ex. 10. Truss given in Fig. 18. Solve for designated loads.

Ex. 11. Fig. 19. Solve for designated loads, with or without additional supports at the ends.

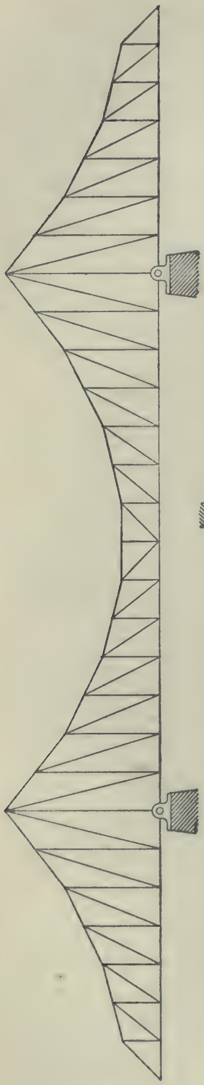

FIG. 19.

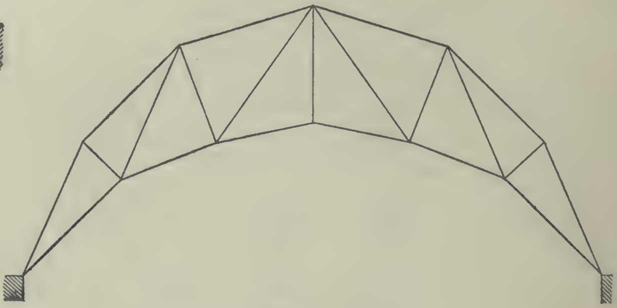

FIG. 18.

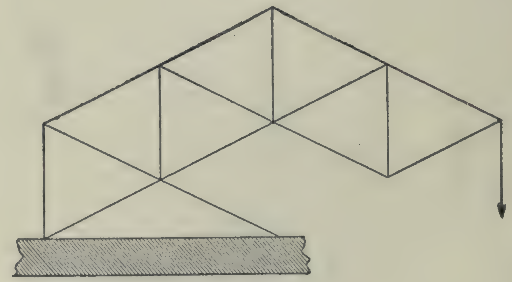

FIG. 20.

Ex. 12. Fig. 20. Solve for the given load. Also find the reactions directly by the parallelogram of forces.

Ex. 13. Fig. 21. In constructing the stress diagram for this figure it will be found that there are three unknown forces at the fourth vertex, and hence the problem is indeterminate by the usual method of closing the force polygon. It then becomes necessary to find the stress on some member of the truss by another method, so that the number of unknowns at the fourth vertex may be reduced to two, thus permitting the closing of the force polygon in the usual manner.

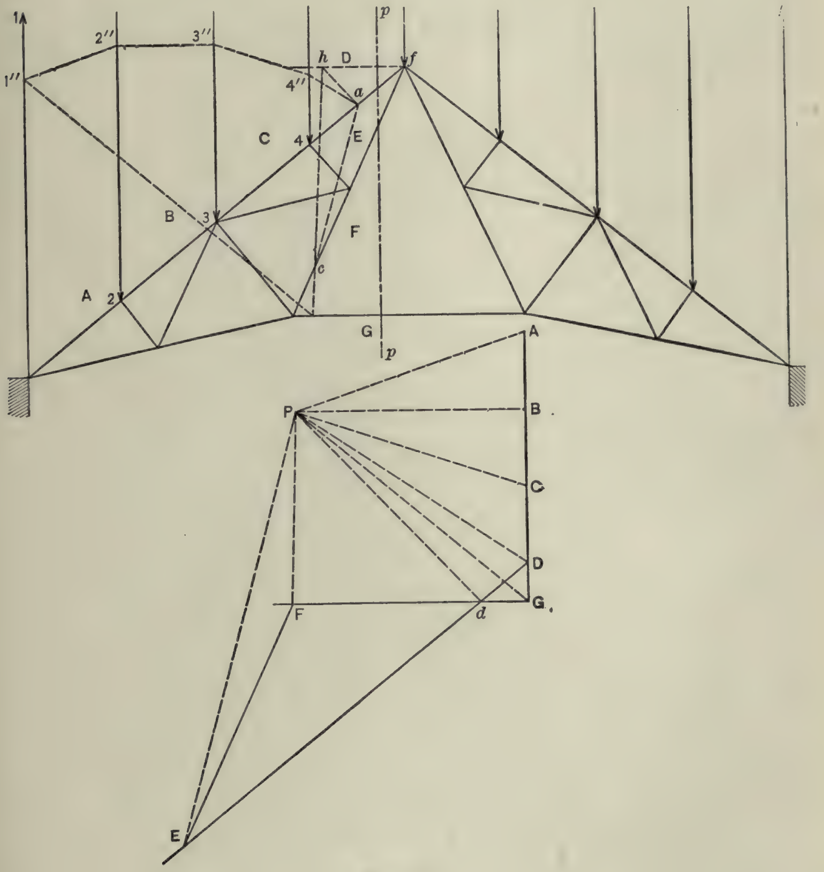

Fig. 21.

The simplest method is to find the stress on *FG* by equating its moment about *f* to the sum of the moments of forces 1, 2, 3, 4. This reduces the unknowns to two at the left end of *FG*, and the stress diagram may then be completed for this vertex.

The problem may also be solved by the method of finding three unknown non-concurrent forces, either directly or by the graphical method described in Art. 10. The plane *pp* cuts the members *DE*, *EF*, and *FG*. The stresses in these members hold the forces 1, 2, 3, 4, in equilibrium, and we may find these stresses by constructing the force polygon *CABCDEFG*. The accuracy of the construction is tested by the parallelism of *ca* and *EP*, which are the final lines in the construction of the reciprocal figures whose vertices are *hfca* and *PdEF*.

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PROBLEMS IN PHYSICS

Used at Worcester Polytechnic Institute *

MECHANICS AND PROPERTIES OF MATTER

Velocity and Acceleration

1. A train moves from rest with constant acceleration. If it travels the first 2 miles in four minutes, in what time will it travel the next 3 miles?

2. Two bodies 1 mile apart begin to move toward each other, one with an initial velocity of 20 ft. per second and an acceleration of 10 ft. per second^2† and the other with an initial velocity of 30 ft. per second and an acceleration of 20 ft. per second^2. When will they meet?

3. A body is thrown vertically upward with a velocity of 150 ft. per second. At what height will its velocity be 50 ft. per second? When will it be at a height of 200 ft.?

4. A stone is thrown upward with a velocity of 150 ft. per second, and one second later another stone is projected with a velocity of 200 ft. per second. When and where will the stones meet?

5. A body is projected vertically upward with a velocity of 200 ft. per second. How long and how high will it rise?

6. A body is caused to move with simultaneous velocities of 100 cm. per second and 200 cm. per second and its resultant velocity is 150 cm. per second. What is the angle between the components?

7. Find the resultant of velocities 10, 15, and 20, which make angles of 30°, 0°, and $-45°$ with the horizontal axis.

8. A train rounds a curve of 1000 yd. radius with a velocity of 30 miles per hour and an acceleration along the tangent of 1.5 ft. per second^2. Find the resultant acceleration.

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* These problems are used through the courtesy of Prof. Masius, Ph.D., of Worcester Polytechnic Institute and P. Blakiston's Son & Co.

† Second^2 is used to indicate feet per sec. per sec.

9. From an airship 6440 ft. above the ground and traveling horizontally due west with a velocity of 12 miles per hour a ball is thrown due east with a horizontal velocity of 17.6 ft. per second. When and where will it strike the ground?

10. A ball is projected horizontally at the same instant as another ball is dropped vertically downward. The latter strikes the ground after one second and the former strikes the ground 4 miles away. Find the velocity of projection.

11. A body is projected at an angle of 30° with the horizon with a velocity of 25 miles per second. When and where will it again meet this horizontal plane? How far will it ascend?

12. An aviator is traveling at a level of 400 ft. with a velocity of 60 miles per hour. From what position must he drop a bomb so that it shall fall on the center of the deck of a battleship, which is at rest? When must a cannon ball be fired vertically from the vessel with a velocity of 1000 ft. per second so that it may strike the aviator?

13. A body is thrown into the air at an angle of 45° with a speed of 50 ft. per second. How far is it from the starting-point when its motion is horizontal?

14. A ball thrown by A is caught three seconds later by B who is 108 ft. from A and on the same level. At what inclination and with what velocity was the ball thrown?

15. A body projected horizontally from the top of a tower 100 ft. high on level ground strikes the ground 100 ft. from the foot of the tower. Find magnitude and direction of the velocity of the body when it strikes the ground.

16. A ball is projected at an angle of 60°, and, one second later, is moving at an angle of 45°. How far will it travel horizontally?

Force and Mass

17. (*a*) What force in dynes will move 1 kg. 500 cm. from rest in five seconds? (*b*) What force in pounds weight will move 1000 lb. 50 ft. from rest in five seconds?

18. What force must be applied to a mass of 100 lb. to move it 50 ft. from rest in five seconds, if the frictional resistance be equal to the weight of 2 lb.?

19. A mass of 955 g. is attached to a spring balance on an elevator rising at such a rate that the balance indicates 1120 g. weight. Calculate the acceleration of the elevator.

20. What total outward pressure will a 50-ton locomotive exert when traveling on a curve of 1000 ft. radius at 30 miles per hour?

21. Two component forces are equal and their resultant is $\frac{3}{2}$ either. Find the angle between them.

22. Three forces 4, 6, 8 are in equilibrium. Find the angles between them.

23. Calculate the various actions and reactions when a horse that weighs 1000 lb. starts a load of 2 tons with an acceleration of 5 ft. per second^2, the connecting rope being of 10 lb. mass and sustaining a tension of 100 lb. weight at the forward end.

24. A rope of 16.1 lb. mass, attached to a block of 48.3 lb. mass is pulled forward in a straight line by the application of a force of 5 lb. weight with an acceleration of 2 ft. per second^2. Calculate the magnitude of the force exerted by the rope on the block and the frictional resistance between the block and the ground.

25. A rapid-firing gun delivers in a second 5 projectiles of 1 lb. each with a speed of 1200 ft. per second. What force is required to hold the gun at rest?

26. A mass of 10 kg. hanging freely draws a mass of 8 kg. up a smooth plane whose inclination is 30°. Find the acceleration and the tension of the string.

27. Masses of 6 g. and 10 g. are connected by a light cord that passes over a frictionless pulley. The former hangs freely while the latter rests on a smooth horizontal plane. Find (*a*) the acceleration, (*b*) the tension on the string.

Work and Energy

28. A small sphere of mass 0.25 lb. moves around the circumference of a circle of 2 ft. radius at a rate of 45 revolutions per minute. What is the central force exerted on the sphere and how much work does this force do in one minute?

29. Calculate the energy (in ft. lb.) required to project a 4-oz. bullet horizontally so that it falls but 1 ft. in 500.

30. Steam is shut off when a train traveling at 10 miles per hour begins a grade of 1 in 100. How long and how far will it ascend, disregarding friction?

31. A body of 1 kg. mass is pushed up 6 meters in 49 seconds on a smooth plane inclined at 30° to the horizontal. Find the work done and the power.

32. A body arrives at the foot of a smooth plane inclined at 30° to the horizontal with a kinetic energy of 9.8 joules and slides up 100 cm. What is its mass?

33. A pendulum consisting of a mass of 5 lb. attached to a very

light cord 1 yd. long is drawn 30° from the vertical and released. Calculate the tension in the cord as it passes through the vertical.

34. A spring has an unstretched length of 6 cm. When supporting a weight of 80 g. it is 8 cm. long. How much work is required to stretch this spring from 10 cm. to 14 cm.?

35. A constant force acting on a mass of 100 lb. gives it a velocity of 20 ft. per second in five seconds. Find the horse-power at which the force works.

36. A mass of 20 kg. hung by a cord 2 m. long is drawn 30° from the vertical and released. It is stopped in a vertical position by a spring which is compressed 2 cm. What force does the compressed spring exert?

37. A water-pumping engine delivers 30 cu.ft. of water per minute. The water is raised 100 ft. and flows off with a speed of 10 ft. per second. At what power must the engine be working?

38. A projectile moving with a speed of 20 m. per second penetrates to the depth of 3 cm., overcoming a constant resisting force. If the velocity were doubled, to what depth would it penetrate?

Rotation

39. A wheel of radius 10 cm. revolving at the rate of 1000 revolutions per minute is brought to rest in one minute. How far does a point on the circumference of the wheel travel before coming to rest?

40. A log rolls on a plane. At a given instant, what point has the largest linear speed and what one has the smallest?

41. A wheel rolls on the circumference of a circle. What vector would represent the instantaneous angular velocity?

Center of Mass

42. A circle has a diameter of 18 cm. A smaller circle tangent to it and 12 cm. in diameter is cut out of it. Where is the center of mass of the remainder?

43. Four similar small heavy balls are placed at the corners of a tetrahedron. Where is their center of mass?

44. Masses of 1, 3, 4, and 12 lb. are placed at the corners of a square of 2-ft. edge. Find the distance of the center of mass from the 1 lb.

45. Two spheres of radii 6 and 8 cm. respectively are joined along a diameter. Where is the center of mass?

46. A thin wire is bent to the form of a triangle. Let A, B, C be the centers of the three sides. Prove that the lines bisecting the

angles of the small triangle *ABC* will intersect at the center of mass of the wire frame.

Moments

47. A circular disk of 500 g. mass and 10 cm. radius is revolving about a normal axis through its center at the rate of two revolutions per second. What is the smallest force that will stop it in ten seconds?

48. A cylinder of mass 100 g. and radius 10 cm. is set into rotation about its geometrical axis by a constant couple. If after ten seconds it has acquired an angular velocity of 5 revolutions per minute, and the forces constituting the couple are applied tangentially to the cylinder, how large must they be?

49. A disk of 1000 g. mass and 20 cm. radius is mounted on an axle of 1 cm. radius and allowed to roll on the axle down a plane inclined at 30° to the horizontal. Calculate (*a*) angular, (*b*) linear acceleration (disregard moment of inertia of axle, which is very small). (*c*) Why from the point of view of energy is the linear acceleration so small?

50. Prove that spheres of different mass and different radii roll down an inclined plane with the same linear acceleration.

51. Prove that a sphere rolls down an inclined plane with a greater linear acceleration than a cylinder.

52. What is the kinetic energy of a sphere of 5 cm. radius and 1 kg. mass, if its center moves with a speed of 10 m. per second and if it rotates about the center 20 times per second?

53. A hoop of 15 cm. radius and 200 g. mass hangs on a nail. Find its moment of inertia about the nail.

54. What is the least force that will prevent a barrel of 2 ft. radius and 1 ton mass, resting on a rough plane inclined at 30° to the horizontal, from rolling down the plane?

Machines

55. Two children of 60 and 90 lb. weight sit at opposite ends of a board 1 ft. from the end. The board is 12 ft. long and weighs 40 lb. Where should it be supported to be in equilibrium?

56. A body weighs 10.2 g. when weighed in the left pan and 10.4 g. when weighed in the right pan of a balance. Find (*a*) the ratio of the balance arms, (*b*) the true weight of the body.

57. If a load of 1000 lb. were hung from an axle of 2 in. diameter how large a force acting on a rope passing over a wheel of 10 in. diameter would be in equilibrium with it?

58. If the lever of a jack screw is 3 ft. long and the screw threads are two to the inch, what force must be applied to the handle to lift 1 ton, supposing friction negligible.

59. (*a*) A body is pushed up a smooth plane inclined at 30° to the horizontal by a horizontal force P or a force F parallel to the plane. Find the ratio of F to P. (*b*) If the plane were not smooth, i.e., if friction not negligible, would the ratio of F to P be different? Why?

Parallel Forces

60. A plank (length 12 m.) rests on a cliff with one-third of its length projecting over the edge. An animal, whose weight is $\frac{3}{4}$ of that of the plank, walks out on it. When will the equilibrium be disturbed?

61. Three parallel forces lie in one plane and are in the direction of the y axis of the coordinate system, the first two of magnitude 15 and 45, respectively, are directed downward and the third of magnitude 30 is directed upward. The coordinates of the points of action are 3, -3; 9, -3; 11, 0.

Find (*a*) the magnitude of the resultant, (*b*) the point of action of the resultant, (*c*) the resultant moment about an axis through $P(+1, -1)$ at right angles to the plane of the forces.

62. To one end of a uniform board of 20 lb. weight and 10 ft. length a vertical string is attached which, after passing over a pulley, is connected to a weight of 5 lb. The board is kept in equilibrium in a horizontal position by one other force F. Find direction, magnitude and point of application of F.

63. A uniform board of 20 lb. weight and 10 ft. length is free to rotate about a hinge at one end. Three feet from this end a weight of 20 lb. is hung from the board. What is the least force that will keep it at rest in a horizontal position, and where and in what direction must it be applied?

64. Parallel forces 2, -4, 6, -8 act at the corners of a square. Find the point of action of the resultant.

Equilibrium

65. A spherical cannon ball that weighs 60 lb. rests between two smooth planes inclined 60° and 30° respectively to the horizontal, making an angle of 90° with each other. Calculate the force it exerts on each.

66. A string, 150 cm. long, is supported from two pegs 1 m. apart

and on the same level, and carries a weight of 2 kg. at the middle. What is the tension of the string?

67. A mass of 10 lb. hangs from a cord 3 ft. long. (*a*) What horizontal force will draw it out 1 ft. from the vertical? (*b*) What will then be the tension in the supporting cord?

68. A uniform board, 1 m. long, rests on a smooth horizontal rail and its lower end presses against a smooth vertical wall making an angle of 45° with it. Where must the rail be placed to keep the board in equilibrium?

Periodic Motions

69. A body of 100 g. mass is performing simple harmonic motion with an amplitude of 1 cm. and a maximum acceleration of 4 cm. per second^2. Find (*a*) its period, (*b*) its kinetic energy when passing through the center of the path.

70. A body of 1 kg. mass performs simple harmonic vibrations in a period of 2 seconds and with an amplitude of 5 cm. What is the magnitude of its acceleration and that of the force acting on it when its displacement is 1 cm.? What is its potential energy at the end of the path and its velocity at the middle?

71. A body attached to a wire as a torsion pendulum makes 12 vibrations per minute. When a sphere of 1 kg. mass and 10 cm. radius is rigidly attached below the body so that its center is in the line of the wire the number of vibrations per minute is 8. Calculate the moment of inertia of the first body.

72. What is the period of vibration of a uniform rod 1 m. long about a point in the rod 10 cm. from one end?

73. The rim of a disk of 200 g. mass and 5 cm. radius is attached to the lower end of a rod of 50 g. mass and 20 cm. length. Find the period of this pendulum when it vibrates about the upper end of the rod.

74. What is the period of vibration of a disk of 10 cm. radius when it vibrates about a nail near the rim?

75. A simple pendulum of 1 m. length makes 239 vibrations in eight minutes. What is the length of a simple seconds pendulum at the point?

76. A rod of 1 m. length is freely supported at one end. Calculate its period of vibration.

Friction

77. A horse exerting a force of 100 lb. weight draws a load of one ton on a sled, which weighs 500 lb., the distance of one mile. Cal-

culate the coefficient of friction, the work done by the horse, and the horse-power exerted by the horse.

78. A block weighing 50 kg. slides on a plane inclined at 35° to the horizontal, the coefficient of friction being 0.25. What pull upward parallel to the plane will just allow the block to (*a*) slide down at a uniform rate, (*b*) move up at a uniform rate, (*c*) ascend with an acceleration of 20 cm. per second per second?

79. (*a*) State a formula for the force in pounds weight exerted by an engine when it is drawing a train of M lb. mass up an incline of $i°$ with an acceleration a, the coefficient of friction being n. (*b*) Derive a formula for calculating the horse-power at which the engine is working at the moment when the speed is v.

80. A spiral spring is attached horizontally to a block of 10 lb. lying on a table. The spring is stretched 10 in. before the block starts. Calculate the work done in stretching the spring, the coefficient of friction being 0.25.

81. How far will a body slide up a plane inclined at 45° to the horizontal, if the initial velocity is 80 meters per second and the coefficient of friction 0.05?

Gravitation

82. Where would a body be attracted by the earth and the moon with the same force? (The mass of the moon is $\frac{1}{80}$ of that of the earth.)

83. How large is the percentage loss in weight of a body on the equator due to the rotation of the earth? (The radius of the earth is about 4000 miles.)

84. Two planets move around the sun in (approximately) circular orbits. Prove that the squares of their periods of revolution are proportional to the cubes of the radii of their orbits.

Elasticity

85. Find the weight required to stretch a steel wire 1 m. long and of 0.5 mm. diameter.

86. To what pressure must an iron sphere be subjected in order to decrease its volume by 0.01 per cent?

87. A cubical block of jelly of 25 cm. edge has two parallel and opposite forces of 1 kg. each applied to opposite faces. If the shear is 0.02, calculate the relative displacement and the shear modulus.

88. A sphere of 200 g. mass impinges directly with a velocity of 100 cm./sec., on a sphere of 100 g. mass at rest. If the coefficient

of restitution is 0.80 find (*a*) velocities after impact, (*b*) changes of momentum and kinetic energy.

89. Two bodies with masses as 2 : 4 are moving toward each other in a straight line with speeds of 10 and 20 ft. per second, respectively. They impinge, the stress during impact being in the line of motion. Find the velocities after impact, the coefficient of restitution being 0.4.

90. A ball falls from a height of 16 ft. and rebounds from a stone slab to a height of 8 ft. Find the coefficient of restitution.

Properties of Liquids

91. A body weighs 500 g. in air, 400 g. in water and 420 g. in alcohol. Find the specific gravity of alcohol.

92. A hollow spherical shell of iron just floats without sinking when completely immersed in water. If the external radius is 10 cm., how large is the internal radius?

93. A rectangular block consists of wood of specific gravity 0.7 and of lead. It just floats when completely immersed in water. Find the ratio of the volumes of the wood and the lead.

94. A "gold" crown weighs 10 kg. in air and loses 627 g. in weight when weighed under water. How much gold and how much silver does it contain?

95. If a submarine weighs 100 tons and displaces 6000 cu.ft. when immersed, how much water must it take in to sink?

96. What is the smallest number of cubic feet of wood of specific gravity 0.7 that a raft must contain in order to carry the weight of 100 lb., when it floats on water of specific gravity 1.03?

97. A cylindrical can 20 cm. in diameter and 10 cm. deep weighs 500 g. If placed on a tank of water, what depth of mercury will it hold without sinking?

98. In order to raise a ship sunk in shallow water 30 empty watertight barrels each of 75 cu.ft. volume and 500 lb. weight are attached to it and the ship just begins to rise. Find the weight of the ship in water (specific gravity of water 1.03).

99. How many grams of water will be drawn up a tube of 1 mm. diameter?

100. A shaft 2 cm. in diameter and 20 cm. long revolves five times second inside a cylinder whose internal diameter is 2.1 cm. The two are separated by a uniform film of liquid whose coefficient of viscosity is 4. Find (*a*) rate of shear, (*b*) shearing stress, (*c*) moment which must be applied to shaft.

101. Two solid cylinders of radii 2 and 5 cm. are rigidly joined along their axis, which is horizontal. The smaller one slides as a piston in a circular opening in the wall of an open water tank; the opening is 20 m. below the level of the water and the barometer reads 75 cm. If the larger cylinder also slides as a piston in a vessel containing liquid what pressure must this liquid exert to keep the piston in equilibrium? (Principle of a compression pump.)

102. A retaining wall is 10 ft. wide and 20 ft. long and inclined at 30° to the horizontal. Find in pounds weight the total thrust on the wall when it is just covered by water of specific gravity 1.02.

103. (*a*) Find the outward force exerted by the water on the walls of a circular tank 2 m. in diameter and 2 m. high. (*b*) Find the thrust on the bottom of this tank.

104. The pressure at the bottom of a lake 38 m. deep is three times as large as the pressure at a depth of 6 m. What is the pressure on the surface?

Properties of Gases

105. The materials of which a balloon is made weigh 200 kg. What weight can it lift when it contains 400 cu. m. hydrogen?

106. (*a*) When a volume of 200 c.c. of gas is connected to an open-tube manometer, the mercury stands 10 cm. lower in the open tube than in the one connected to the gas, the barometer reading 74 cm. What is the pressure? (*b*) If the open tube is raised until the mercury stands 5 cm. higher in it than in the other, what volume must the gas occupy to be in equilibrium with the mercury, the temperature remaining constant?

107. Find the pressure which 10 g. of air will exert when placed in a vessel of 10 liters capacity.

108. A vessel containing 20 g. of oxygen at a pressure of 10 lb. per square inch is put in communication with another containing 5 g. of hydrogen at 30 lb. per square inch. What is the resultant pressure?

109. One wall of a room is 9×24 ft. If the room is closed tightly when the barometer reads 30 in., what pressure must the wall sustain when the barometer falls to 29 in.?

110. A tube 2 ft. long is half filled with water and then inverted so that the open end just dips beneath the surface of water in a vessel. How high does the water stand in the tube? (Height of mercury barometer is 74 cm., which may, if convenient, be assumed as equivalent to a water-barometer reading 33 ft.)

111. To what depth must the top of a cylindrical diving bell 250 cm. high be immersed, so that the water (specific gravity 1.026) rises 100 cm. in the bell?

WAVE MOTION

Simple Harmonic Motion

112. A mass of 294 g. is suspended by an elastic band of such elasticity that an additional weight of 15 g. will stretch it 2 cm. It is extended 2 cm. and released. Find the period, and the displacement, velocity and acceleration six seconds after it passes upward through its resting-point.

113. Compound two simple harmonic motions of the same period and in the same plane with amplitudes 4 and 5 and with a phase difference of one-eighth of a period.

114. Compound two simple harmonic motions at right angles with amplitudes in the ratio 1 : 2, periods in the ratio 1 : 2, and starting in the same phase.

115. Two simple harmonic motions at right angles to each other start with no phase difference and have equal amplitudes. Derive the equation of the curve described and prove that it is a circle.

Waves

116. Compound two wave trains of equal amplitude and equal length, starting with a difference of one-quarter of one wave length.

117. Draw a figure showing an incident wave together with its reflected wave (reflection occurring at a fixed boundary) one-fourth, one-half, three-fourths, and one period after reflection.

118. An iron wire of 2 m. length and 1 mm. diameter is stretched by a weight of 2 kg. Find the velocity of a transverse wave in it.

119. Find the velocity of a longitudinal wave in the same wire.

HEAT

Temperature

120. Change the following temperatures on the Fahrenheit scale to temperatures on the Centigrade scale: +105, +67, −10.

121. Change the following temperatures on the Centigrade scale to the Fahrenheit scale: −80, +300, +40.

Expansion

122. A bar of copper is 0.034 in. longer at 25° C. than at 5°. Its coefficient of linear expansion is 0.000017. What is its true length at 0° C.?

123. A glass tube is 80.016 cm. long at 0° C. and 80.096 cm. at 100°. A thin flask of this glass holds 100 c.c. of mercury at 0°. 1.1 c.c. are expelled at 70°. What is the coefficient of expansion of mercury?

124. (*a*) A flask of crown glass contains 500 c.c. at 0°. What is its capacity at 30°, if the coefficient of linear expansion of this glass is 8.8×10^{-6}? (*b*) How many grams of alcohol will it contain at this temperature? (Density of alcohol at 0° is equal to 0.810, coefficient of cubical expansion 0.00105.)

125. A barometer consisting of mercury in glass and provided with a brass scale reads 75 cm. If the temperature of the room is 20°, what is the pressure reduced to 0°?

126. What pressure is exerted by 100 g. of air at 20° C. in a tire of 20 liters capacity?

127. Find the mass in grams of the air in a lecture room at 25° and 74 cm., the dimensions of the room being 17 m. × 17 m. × 4 m.

128. The volume occupied by air in a constant-pressure air thermometer is 50 c.c. in ice water and 68.3 c.c. in steam at 76 cm. pressure. What is the temperature when the volume is 60 c.c.?

Calorimetry

129. What will be the final temperature of a nickel calorimeter of 90 g. mass containing 300 g. of water at 20°, if 750 g. of iron at 90° are dropped into it?

130. Five hundred g. of copper at 98° are dropped into 330 g. of water at 18° contained in a copper can of 80 g. If the final temperature is 28°, what is the specific heat of copper?

Mechanical Equivalent of Heat

131. Two hollow conical pieces of brass, so-called "friction cones," are fitted tightly into each other and the lower one is then rotated 1000 times. The upper one is rigidly connected to a disk of 10 cm. radius which is prevented from rotating by a weight of 100 g. attached to a string passing around the disk. The mass of the two cones together is 250 g. and they contain 40 g. of water. Compute the change in temperature of the water.

132. What would be the rise of temperature of a piece of iron weighing 500 g. when struck by a hammer weighing 20 kg. and having a velocity of 10 m. per second if it absorbs half of the heat produced?

133. A lead bullet traveling with a velocity of 500 m. per second strikes a target and is just melted. What fraction of the heat generated was received by the bullet. (The temperature at impact is 20°.)

134. A wheel of 50 kg. mass and 50 cm. radius turning twice per second is brought to rest by a brake. Calculate the heat developed.

Change of State

135. Calculate the heat required to change 50 g. of ice at $-10°$ to steam at 140°.

136. Steam at 110° C. passes into a condenser through which cold water is flowing. The temperature of the inflowing water is 5.8°, that of the outflowing water 37.6°; the mass of the condensed water is 367 g. and its final temperature 20°. If 7170 g. of water have passed through the condenser, what is the heat of condensation of steam?

137. How much steam passed into a mixture of 5 kg. of ice and 5 kg. of water will bring the whole to a temperature of 50° C., assuming the steam to be originally at 100° C.?

138. Find the relative humidity of air at 20° if the dew point is found to be 10°.

139. A closed vessel contains water, water vapor, and air at 20° C., the pressure being 75 cm. Hg. When the temperature is raised to 100° C. what will the pressure be? Will the water boil at 100°?

140. In a continuous-flow combustion calorimeter 5 liters of gas are burned while 7 liters of water flow through the calorimeter, entering at 10° and leaving at 14°. Find in calories per liter the heat of combustion.

141. Assuming that one liter of the gas in Problem 140 weighs just 1 g., change the result to B.T.U. per pound.

142. A bulb with a graduated stem is filled with cracked ice and water to the 200 c.c. mark. The bulb is immersed in 700 g. alcohol and while the temperature of the alcohol sinks 21° the water drops to the 191 c.c. mark. Calculate the specific heat of alcohol (neglecting the contraction of the bulb).

Heat Conduction

143. The top of a steam chamber at 100° is of stone 70 cm.×60 cm. and 20 cm. thick. If ice is piled on the stone and 3 kg. are melted per hour what is the thermal conductivity of the stone?

144. How many grams of steam will be condensed in 1 hour on an iron pipe of 2 cm. mean radius and 2 mm. thick, 10 cm. length of which is in a steam chamber filled with steam at 100°, if water at 20° flows continuously through the pipe?

145. Two rectangular slabs, each 5×6 cm. wide and 50 cm. long, of concrete and glass respectively are joined at their ends. If the free end of the concrete is kept at 30° and that of the glass at 10°, at what temperature will the junction be, when a stationary flow of heat is established?

Radiation

146. How large a percentage error would be committed by replacing Stefan's law of radiaion by the simplified form (Eq. 2 in §337, Duff) if the temperatures in question were 500° and 495° absolute?

147. Compare the amount of heat radiated between two black surfaces 1 sq. m. in area each and 1 m. apart, with the amount they would exchange by conduction if the space between them were filled by the best conducting substance, silver, when their temperatures are 0° and 100° C.

Thermodynamics

148. Ten liters of dry air are under a pressure of 2 atmospheres. To what volume will they expand when the pressure is suddenly reduced to 1 atmosphere?

149. Thirty-two g. of oxygen at 0° C. and occupying a volume of 22.4 liters are heated at constant pressure to 100° C. (*a*) Calculate the work done. (*b*) What value would follow from these data for the specific heat at constant volume if oxygen were an ideal gas?

150. A gas is compressed from a volume v_1 to a volume v_2. Compare graphically the work required for adiabatic and for isothermal compression and state which is larger and why?

151. The original values of pressure, volume, and temperature of a gas are p_1, v_1, T_1. The gas expands very quickly to a new state p_2, v_2, T_2. The volume is now kept constant and the gas heated to a state p_3, v_3, T_3 such that the final temperature is equal to the original one. (*a*) Plot these changes in a p–v diagram. (*b*) Establish an

equation between the pressure and volume of the first and second state, also an equation between the pressure and volume of the first and third states, and from them deduce a value for the ratio of the specific heats of the gas in terms of pressure only.

152. What would be the efficiency of a Carnot's cycle between 100° and 150° C.?

153. Assuming a boiler efficiency of 60 per cent, at what H.P. would the best steam engines work, when burning 100 lb. of anthracite coal per hour? (See table on page 228, Duff.)

154. The engines of the steamship George Washington are of the quadruple expansion type and deliver 20,000 H.P. Assuming a boiler efficiency of 75 per cent, how many tons of anthracite coal must the steamer carry for an six-day voyage from Liverpool to New York?

Problems in Electricity omitted to Prob. 221.*

SOUND

Velocity of Sound

221. If a projectile is fired upward with a velocity of 400 m. per second, at what height would the projectile arrive at the same time as the sound of the explosion?

222. If the velocity of sound is 340 m. per second at 16° C., what is its value at 53° C.?

223. The density of a certain gas under standard conditions is 0.002 g. per c.c. and the ratio of its specific heats is 1.3. Find the velocity of sound in this gas when the barometer stands at 74 cm. and the thermometer at 30° C.

Doppler's Principle

224. With a velocity of how many miles per hour should an observer be moving toward a source of sound of definite pitch in order to increase the pitch one-half tone (B to C)?

225. Find an expression for the apparent pitch heard by a stationary observer when a source of definite pitch is approaching him. Prove that if the velocity of the source is small compared with the velocity of sound, this expression is practically the same as the one used when the source is at rest and the observer moving. Would a drift of the medium propagating the waves further change the apparent pitch?

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* Electricity taught in another department.

Musical Scales

226. The lowest tone of the human voice is about E_{-1}; and the highest about C_3. Assuming the frequency of A_1 as 435 vibrations per second, calculate the frequencies of these two notes on the equally tempered scale.

Vibration of Bodies

227. A tuning-fork vibrating faster than a standard tuning-fork of frequency 256 gives 2 beats per second when sounded with the standard. A closed pipe of variable length resonates to this fork when adjusted to a length of 33 cm. Calculate the velocity of sound.

228. The second overtone of an open organ pipe is of the same pitch as the first overtone of a closed pipe. If the open pipe is 100 cm. long, what is the length of the closed pipe?

229. Find the wave length of the fundamental and the first overtone of a pipe 1 m. long closed at one end. Calculate the pitch of each note at 16° C.

230. The dust piles in a Kundt's tube were 15 cm. apart in dry air at 18° C. and 12.5 cm. in CO_2 gas. What is the velocity of sound in the latter?

231. An aluminum wire of 1 m. length and 1 mm. diameter is stretched by 2 kg. How large a weight must be used to stretch a similar wire in order to produce 2 beats per second when the two wires are sounded together?

LIGHT

Reflection

232. Find position, nature, and size of the image of an object of 2 cm. length placed (*a*) 60 cm., (*b*) 40 cm. from a concave mirror of 100-cm. radius.

233. Find the position, nature, and size of the image when an object 3 cm. long is placed 20 cm. from a convex mirror of 30-cm. radius. Construct the image graphically.

234. A bright object is 1 m. from a wall and it is desired to throw its image (magnified four times) upon this wall. What is the kind of mirror required, its radius of curvature, and its distance from the wall?

235. With a given concave mirror, where must an object be placed so that the image may be (*a*) real and twice the size of the object? (*b*) Virtual and three times the object in size?

236. Prove that the image formed by a convex mirror is always smaller than the object.

237. How far must a man stand from a concave mirror of 150-cm. radius to see an image of his face of twice its real size?

238. When a dark liquid like coffee in a porcelain cup is placed near a lamp a bright curve of peculiar shape is seen on its surface. Explain.

239. It is desired to show the identity of heat and light rays in a lecture experiment by converging the solar rays to a focus and igniting an inflammable substance placed at the focus. What kind of a mirror should be chosen and why?

240. How large a radius must a concave mirror have in order to form an image of 1-mm. diameter of a planet, the angular size of which is $1'$?

Refraction and Dispersion

241. How far will it be necessary to raise a microscope if a cover glass 2 mm. thick of index of refraction 1.5 is introduced above an object?

242. Derive an expression for the lateral displacement of a light ray passing through a plate with parallel sides in terms of the thickness of the plate, the angle of incidence, and the index of refraction of the substance.

243. What is the angle of a prism of heavy flint glass that will achromatize an 8° crown glass prism for the C and F lines? (Compare the table of indices of refraction on page 588, Duff.)

Glass

244. A narrow pencil of white light is incident at an angle of 45° on a small glass prism with an angle of 60°. If the index of refraction of the C line is 1.66 and that of the F line 1.67, find the width of the spectrum between the C and F line on a screen 1 m. from the prism.

245. White light passes through a wide slit and then through a lens from which the rays emerge parallel. If this light falls on a prism and the refracted light is then thrown on a screen a white image with only the edges colored is formed on the screen instead of a spectrum. Explain this with the aid of a sketch and point out which colors should appear on which sides of the image.

Lenses

246. Find position, nature, and size of the image of an object 2 cm. high, placed 25 cm. from a divergent lens of 15 cm. focal length. Construct it graphically.

247. Find position, nature, and size of the image of an object 3 cm. high formed by a converging lens of 30 cm. focal length when the distance from the object to the lens is (*a*) 40 cm., (*b*) 20 cm.

248. An object is placed 6 ft. from a lens; find the focal length when its image is formed 18 in. from the lens (*a*) on the same side, (*b*) on the opposite side.

249. A concave meniscus (convexo-concave lens) has radii of curvature 25 cm. and 15 cm. (*a*) Calculate the position of the image of an arrow placed 20 cm. from this lens. (*b*) Is it real or virtual, upright or inverted? (*c*) Prove results by geometrical construction. (The index of refraction of the glass is 1.6.)

250. The refractive index of a certain kind of glass is 1.5 for red light and 1.6 for violet light. A plano-convex lens is constructed of this glass, the radius of the curved face being 25 cm. Find (*a*) focal length for red and for violet light, (*b*) linear separation of the red and violet portions of the image of an object 30 cm. from the lens.

251. A concavo-convex (see §679, Duff) lens has radii of curvature of 20 and 30 cm. respectively. Find the position, size, and nature (whether real or virtual, upright or inverted) of the image of an object 1 cm. high and 50 cm. from the lens ($n=1.6$).

252. A plano-convex lens ($n=1.5$) forms a real image of an object placed 2 cm. from it and magnifies it three times. Find the radius of the convex face.

253. An object of 2 cm. length is 15 cm. from a concave lens of 20 cm. focal length; 10 cm. behind this is a convex lens of 15 cm. focal length. Find the position and size of the final image.

254. An object of 1 cm. length is placed 36 cm. from a convex lens of 15 cm. focal length. A second convex lens of 20 cm. focal length is placed behind the first. Find the position and size of the final image (*a*) when the second lens is 47 cm. from the first, (*b*) when it is 15 from the first.

255. The focal length of the object glass of an opera glass is 6 in. A divergent eyepiece of 2 in. focal length gives a virtual image 10 in. from the eyepiece of an object 5 ft. high and 30 ft. from the object glass. Find (*a*) location of image formed by object glass, (*b*) distance between object glass and eyepiece, (*c*) size of virtual image, (*d*) magnifying power of instrument.

256. A candle-flame 3 in. high is 5 in. in front of double convex lens of focal length 4 in.; 28 in. behind this lens is a concave mirror or radius 14 in. Find size, location and character of the final image.

257. An achromatic lens is to be made of a combination of a crown-glass double convex lens ($n_c = 1.56$, $n_f = 1.57$) and a plano-concave flint-glass lens ($n_c = 1.65$, $n_f = 1.67$), the adjacent faces to fit together and the focal length to be 40 cm. Calculate the radii of curvature of the faces.

Photometry

258. A screen is equally illuminated on both sides when it is 3 m. from an arc light and 20 cm. from an incandescent lamp of 100 candle-power. Find the candle-power of the arc light.

259. A lamp is placed on the axis of a concave mirror of 20 cm. focal length; 1 m. from this lamp there is a screen. If the intensity on the screen is four times as large as it would be without the mirror how far is the mirror from the lamp?

Total Reflection

260. Given the index of refraction of glass with respect to air as 1.5 and that of water with respect to air as $\frac{4}{3}$. Is it possible to get total reflection in going (*a*) from water to glass? (*b*) from glass to water?

261. The index of refraction of water is 1.33 and that of carbon bisulfide is 1.67. What is the critical angle between the two?

262. The critical angle of a solid in contact with a liquid is 60°. The critical angle of this liquid in contact with air is 45°. What is the index of refraction of the solid?

Interference

263. What is the thickness of the thinnest soap bubble films that can be determined from the colors of thin plates?

264. A plano-convex lens of 7 m. radius is pressed firmly against a flat surface and illuminated from above by monochromatic light. The twenty-fourth ring has a radius of 1 cm. What is the wave length of the light?

265. The angular deviation of the fourth order spectrum of a certain color is found to be 30°. If the grating contains 212 lines to a millimeter, what is the wave length of the color?

266. A transmission diffraction grating has 100 lines to the millimeter. How far will the first order spectrum of sodium (Fraun-

hofer D line) be from the central bright line on a screen 5 m. distant from the grating?

Diffraction

267. Monochromatic light of wave length 500 $\mu\mu$ and with a plane wave front falls normally on a screen. On the screen is a pin-hole of 1 mm. diameter. How far should a second screen, parallel to the first, be placed from it, so that the intensity of light at a point P on the axis of the pin-hole is a maximum? When the second screen is shifted slowly to a distance ten times as large as the first, how often would the intensity at P change from a maximum to a minimum?

268. Two narrow slits are 0.2 mm. apart. What must be the diameter of a lens 4 m. away to clearly resolve the image of the two slits?

269. How far from a narrow slit of 0.5 mm. width must a screen be placed to give a central diffraction band of 1 mm. width (*a*) if illuminated by sodium light, (*b*) if illuminated by red light (B line)?

Polarization

270. Light falls on a Nicol's prism as in Fig. 586, Duff. Calculate the angle at which the ordinary ray falls on the Canada balsam surface. Calculate also what this angle would be if the Nicol were not ground off before splitting it, and from these two results deduce certain limits between which the index of refraction of Canada balsam with respect to Iceland spar must lie.

271. How thick should a crystal of Iceland spar cut with faces parallel to the axis be, so that the distance between the two images of a mark observed normally through the crystal will appear to be 0.5 cm.?

272. A column of cane sugar is 20 cm. long and rotates the plane of polarization of sodium light through 62°. What is the concentration of the solution?

Spectrum

273. How high must the temperature of a body be when the maximum intensity is in the range of the visible spectrum (red B line)?

274. The motion of Sirius has been found by means of Doppler's principle, from observations of the hydrogen F line, to be 66 km. per second from the earth. What shift of this line has been observed?

MECHANICS AND FLUIDS

PROBLEMS, USED AT RENSSELAER POLYTECHNIC INSTITUTE *

VALUE OF CONSTANTS USED IN THE PROBLEMS

1 in. = 2.540 cm.

1 lb. = 453.6 g.

"g" = 980 cm. per second per second = 32.2 ft. per second per second.

1 H.P. = 746 watts.

Absolute zero = $-273^\circ.07$ C. = $-459^\circ.5$ F.

1 calorie = 4.19 joules.

1 B.T.U. = 778 ft.-lb.

Latent heat of water = 80 calories. Latent heat of steam = 536 calories = 966 B.T.U.

Velocity of light = 3.0×10^{10} cm. per second = 186400 miles per second.

Wave length of sodium light = 5893×10^{-8} cm.

Index of refraction of water = 4/3.

Velocity of sound = 33200 cm. per second = 1090 ft. per second at 0° C.

1. A body has a velocity of 60 miles per hour. Find its velocity in feet per second and in centimeters per second.

88 ft. per second; 2684 cm. per second.

2. A body is in equilibrium under the influence of three forces. Two of these forces act in directions inclined to one another at 60° and their magnitudes are 40 dynes and 30 dynes respectively. Find the magnitude and direction of the third force.

154°.7 with the greater force; 60.83 dynes.

3. A man desires to cross a river 1800 ft. wide. The current flows at the rate of 4 miles per hour. In what direction must he row in order to land directly opposite his starting point, and how long will it take him? He can row in still water 5 miles per hour.

Upstream 53°.12; 6.82 minutes.

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4. A mass of 600 g. starts from rest and in six seconds has a velocity of 36 m. per second. Find the force. $F=36\times10^4$ dynes.

5. What is the force of gravity on a body whose mass is 700 lb? 311.2×10^6 dynes.

6. A body is thrown vertically upward with a velocity of 697 cm. per second. How long will it take to rise 195 cm. and what velocity will it then possess? 0.383 second; 322 cm. per second.

7. How long will it take a body to fall 650 ft. and what velocity will it acquire? 6.35 seconds; 204.6 ft. per second.

8. A particle starts from rest and travels for five seconds with a uniform acceleration of 8 cm. per second per second then continues moving at the uniform speed acquired for ten seconds, and is finally brought to rest with a uniform acceleration equal to -0.5 cm. per second per second. Calculate the total space passed over and the total time taken. 21×10^2 cm. 95 seconds.

9. Masses of 978 and 762 g. respectively, are hung by a flexible cord passing over a frictionless pulley. How far must the weights move in order to acquire a velocity of 325 cm. per second? 434.1 cm.

10. A mass of 1 kg. connected to a fixed point by a string 1 m. in length, and whirled round in a circle once a second. Find the tension on the string. 3.944×10^6 dynes.

11. A mass of 1 g. executed simple harmonic motion with an amplitude of 4 cm. and a period of 0.5 second. Find the force toward the center when the phase is $\frac{T}{4}$, $\frac{T}{8}$, and $\frac{T}{2}$ respectively.

631.6 dynes; 442.1 dynes; 0.

12. Show that the spaces passed over, starting from the center by a body executing simple harmonic motion, in the successive time intervals $\frac{T}{16}$ are approximately proportional to 4, 3, 2, and 1.

13. The diameter of a cylinder of a steam engine is 18 in. and its length 24 in. What work will be done in each stroke of the piston if the average pressure of the steam is 110 lb. per sq.in.? 0.538×10^5 ft.-lb.

14. A belt transmits power from a wheel $3\frac{1}{2}$ ft. in diameter, which revolves 180 times per minute. The difference between the tensions on the tight and slack sides of the belt is equal to 200 lb. What horse-power does the belt transmit? 12 H.P.

15. Find the horse-power of an engine that should be employed for raising coal from a pit 200 ft. deep, the average daily yield being 1900 tons. 15.99 H.P.

16. Determine the rate at which an engine is working when it

drives a train of 150 tons at a rate of 30 miles per hour, the resistance to motion being equal to 16 lb. per ton. 192 H.P.

17. A weight of 390 lb. is carried by a bar, *AB*., 13.2 ft. long supported at the ends. If the distance of the weight from the end *A* is 5.6 ft., what is the pressure of each support?

A. 225 lb.; *B*. 165 lb.

18. What force acting horizontally can keep a mass of 16 kilos at rest on a smooth, inclined plane, whose height in 3 m. and base 4 m.? 12 kilos.

19. If a mass weighs 29.620 g. in one pan of a balance and 28.710 g. in the other, what is its true mass? 29.165 g.

20. A grindstone of radius 20 cm. and density 2.2 is 10 cm. thick. Calculate its moment of inertia when rotated about an axis through the center. 5529×10^3 g. cm^2.

21. If this stone rotate five times in four minutes, compute its kinetic energy. 47,371 ergs.

22. What is the length of a pendulum which loses twenty seconds a day, where " g " = 980.3 cm. per second per second? 99.37 cm.

23. A compound pendulum is composed of a wire 60 cm. long, whose weight may be neglected, suspended from one end, and carrying masses of 3, 5, and 7 kilos at distances of 20, 40 and 50 cm. from the point of support. Compute its length and time of double vibration, if g. = 980 cm. per second per second. 1.4 seconds.

24. A wire of length 250 cm. and radius 0.3 mm. is stretched by hanging a weight of 12 kilos, and the elongation produced is 8 mm. Calculate the value of Young's modulus for the material of this wire.

13.0×10^{11} dynes.

25. The cylinder of a hydraulic press is 10 in. in diameter. The plunger is forced outwards by means of a small pump which supplies the press cylinder with water, its piston being $\frac{1}{2}$ in. in diameter and its stroke $1\frac{1}{2}$ in. If a force of 100 lb. be applied to the pump piston, how great a force can it exert on the plunger, and how far does the plunger advance for one stroke of the piston?

40,000 lb., 0.00375 in.

26. A glass bulb weighs in air 4.56 g. When immersed in water at 20° C. the weight is 2.53 g. When immersed in alcohol the weight is 2.94 g. The density of water at 20° C. is 0.9983. What is the density of the alcohol? 0.7965.

27. A triangular plate is immersed vertically in water, with the vertex in the surface of the water and the base horizontal. The height and base of the triangle are each 50 cm. Find the force on the face of the plate. 4038×10^4 dynes.

28. If the density of ice be 0.9179, and of sea water 1.025, what portion of an iceberg is above water? 10.4 per cent.

29. A liter of air under normal conditions of temperature and pressure weighs 1.293 g. Find the weight of the air in a liter flask when the barometer stands at 72 cm., the temperature being 0° C. 1.225 g.

30. The mouth of a vertical cylinder 18 in. high is closed by a piston whose weight is 6 lb. and area is 6 sq. in. If the piston is allowed to fall, how far will it descend, supposing the atmospheric pressure to be 14.7 lb. per sq. in. when the piston is inserted? 1.15 in.

31. The inner tube of an automobile tire may be taken as a cylinder, 10 cm. in diameter and 100 cm. long. What volume of air measured at atmospheric pressure must be pumped in to raise the pressure to 5 atmospheres? 39,260 cu. cm.

32. A U tube is partly filled with water. How many inches of oil having a density of 0.79 must be added to raise the water 4.5 in. in one leg above its first level? 11.4 in.

33. A glass tube used for sounding is 38.1 cm. long and open at the lower end. The inside is covered with a soluble pigment, and the tube lowered to the bottom, in sea water, density 1.03. On raising it to the surface it was found that the water had entered the tube to depth of 23.6 cm. Find the depth of the sea water. Barometric reading 74 cm. 1614 cm.

34. A vessel filled with water has a circular orifice 6 cm. in diameter, 298 cm. vertically below the surface of the liquid. If the water be maintained at its initial depth, by supply from without, calculate the theoretical discharge per minute. 129.7×10^4 cu. cm.

35. How high will water stand above hydrostatic level in a tube 0.057 cm. in diameter, assuming the angle of contact to be very small? $T = 81$ dynes per cm. 5.8 cm.

36. What is the pressure within a soap bubble 12.5 cm. in diameter if the surface tension of the liquid be taken as 80 dynes per centimeter, and what is the total force exerted by the film on the gas within? 51.20 dynes per cm.^{2}; 25.14×10^3 dynes.

SOUND, LIGHT AND THERMO-MECHANICS

PROBLEMS, USED AT RENSSELAER POLYTECHNIC INSTITUTE

1. Assuming the velocity of sound in air at 0° C. to be 331.36 m. per second, calculate the velocity in hydrogen at the same temperature, having given the mass of one liter of hydrogen = 0.0896 g. 1259×10^2 cm. per second.

2. A stone is dropped into a well and the sound of the splash is heard after five seconds. How deep is the well, if the temperature be 10° C.? 107.5×10^2 cm.

3. Colladon and Sturm measured the velocity of sound in the waters of Lake Geneva and found that it traveled 1435 m. per second, the temperature being 8°.1 C. Compute the coefficient of elasticity for water at this temperature. 20.59×10^9 dynes per cm^2.

4. How many beats per second will there be between two notes which represent the major third above the C of 256 vibrations per second on the natural, and on the equal tempered scale? 2.46.

5. The prongs of a tuning fork making 256 vibrations per second are placed over the end of a tube which is partly filled with water. The height of the water in the tube is then varied. There is a maximum intensity when the distance from the water level to the top of the tube is 31.1 cm. and also when it is 98.3 cm. The temperature is 18° C. If the velocity of sound increases 60 cm. for each degree rise in temperature, what is the velocity of sound in air at 0° C.?
33,320 cm. per second.

6. An observer listening to the whistle of an engine which is approaching him at the rate of 45 ft. per second notices that the pitch of the note which he hears is the same as that of a tuning fork which makes 458 vibrations per second. What is the actual pitch of the whistle? 440 vibrations per second.

7. Notes of 225 and 336 vibrations per second in which the first and second overtones were present are simultaneously sounded. How many beats will be heard per second? 3.

8. The "e" string of a violin is tuned so as to vibrate 640 times per second. The vibrating portion has a mass of $\frac{1}{8}$ g. and a length of 33 cm. What is the stretching force? 6.759×10^6 dynes.

9. An open organ pipe produces 8 beats per second at 20° C. when

sounded with a tuning fork of 256 vibrations per second, the fork giving the lower note. How much must the length of the pipe be altered to bring it in accord with the fork? Lengthened 2.04 cm.

10. The top of an open organ pipe is suddenly closed. What effect is produced on the pitch and character of the note? If the vibration frequency of the overtones next above the fundamental note in the two cases respectively differ by 256, what was the pitch of the organ pipe? 512 vibrations per second.

11. An open organ pipe 120 cm. in length is tuned correctly when the room temperature is 20° C. What will be the change in its frequency when the temperature rises to 32° C.?

3 vibrations per second.

12. In an experiment with Kundt's tube the length of the rubbed glass rod was 900 cm. and the nodal points were found at the marks 0, 63, 127, 188, 252, on a scale divided in centimeters. What was the velocity of sound in glass? 4743×10^2 cm. per second.

13. What is the temperature 92° F. on the Centigrade scale?
What is the temperature 15° C. on the Fahrenheit scale?
What temperature is expressed by the same number on the Centigrade and Fahrenheit scales?

33°.3 C.; 59°.0 F.; −40°.0.

14. The steel cable from which the Brooklyn Bridge hangs is more than a mile long. How many feet does a mile of its length vary between a winter day when the temperature is −20° C. and a summer day when it is 30° C.? (The coefficient of expansion of steel $= 12.1 \times 10^{-6}$.) 3.19 ft.

15. A clock with a seconds pendulum of iron is correct at 25° C. How many seconds per day will it gain if the temperature is 0° C.? What will be the result if the pendulum be made of nickel-steel? The coefficient of expansion of nickel-steel is 1.0×10^{-6} and of iron is 12.1×10^{-6}. 13.07 seconds; 1.08 seconds.

16. The coefficient of linear expansion of glass is 0.000008 and a certain glass flask contains exactly 100 c.c. of water at 0°, what will be the volume of water contained when the flask and its contents are heated to 100°? 100.24 c.c.

17. At sea level the barometer stands at 750 mm. and the temperature is 7° C., while on top of a mountain the barometer stands at 400 mm. and the temperature is −13° C. Compare the weights of a cubic meter of air in the two places. 1; 0.5744.

18. Find the volume of 3 lb. of nitrogen at a pressure of 60 lb. to the square inch, and a 80° F. The volume of 1 lb. nitrogen at 32° F. = 12.76 cu. ft. 10.29 cu. ft.

19. Suppose 20 cu. cm. of hydrogen gas to have been collected in a closed tube over mercury. The barometer reading corrected for temperature is 74.5 cm. and the temperature of gas is 24° C. The mercury in the tube stands 12 cm. above the mercury in the dish in which the tube is inverted. Reduce the volume of the gas to standard conditions. 15.13 cu. cm.

20. A certain boiler boils dry, containing steam at 80 lb. per sq. in. at 155° C., which in the absence of water may be considered as a perfect gas. If the temperature should rise to 600° C., would the boiler burst, if it can just support a pressure of 300 lb. per square inch.? No. $P=163.2$ lb. per square inch.

21. A mass of 700 g. of copper at 98° C. put into 800 g. of water at 15° C. contained in a copper vessel weighing 200 g. raised the temperature of the water 21° C. Find the specific heat of the copper. 0.091.

22. In order to ascertain the temperature of a furnace a platinum ball weighing 80 g. is introduced into it. When this has attained the temperature of the furnace it is rapidly transferred to a vessel containing 400 g. of water at 15° C. The temperature of the water rises to 20° C. What was the temperature of the furnace? Specific heat of platinum $=0.03$. 853°.4 C.

23. A cannon ball, the mass of which is 100 kg., is projected with a velocity of 500 m. per second. Find the amount of heat which would be produced if the ball were suddenly stopped.

2.983×10^6 calories.

24. If the average pressure in the cylinder of a steam engine be 10 kg. per square centimeter, and the area of the piston be 300 sq. cm., how many calories does the steam lose when it pushes the piston 50 cm. forward? 3508 calories.

25. A heat engine working two temperatures could theoretically convert one-sixth of the heat supplied into useful work, but if the lower temperature were reduced by 65° C. the theoretical efficiency would be doubled. Find the temperatures.

117° C., 52° C.

26. In obtaining work from an engine at the rate of 20 H.P. 56 lb. of coal are consumed per hour. Find the efficiency of the engine, assuming that the heat produced by the combustion of 1 lb. of coal is sufficient to convert 15 lb. of water at 100° into steam at the same temperature? 6.276 per cent.

27. A copper vessel weighs 271 g. and when a quantity of water is added the weight is 1292 g. The temperature of the water is 34°.6 C. A mass of ice is then added which cools the water to 14°.0 C. and the

resulting weight is 1516 g. The specific heat of copper is 0.095. Determine the latent heat of fusion of ice. 80.00 calories.

28. From what height must a lump of ice at 0° C. fall in order to melt itself, supposing it possible to concentrate all the energy of the fall in the lump? 3.421×10^6 cm.

29. 100 cu. cm. of mercury are heated to 110° C. and then poured into a cavity in a block of ice. It is found that 65 c.c. of water are obtained. Taking the density of mercury as 13.6 find its specific heat. 0.03476.

30. Into a calorimeter containing 120 g. of water at 10° C. steam is passed. The total weight of steam condensed is 5 g. and the final temperature of the water in the calorimeter is 35° C. What value does this give for the latent heat of steam? 535 calories.

31. At what temperature will water boil in Denver, 5000 ft. above sea level? 94°.8 C.

32. The latent heat of steam at 100° is 536. If a kilogram of water when converted into saturated steam at atmospheric pressure occupies 1.651 cu. m., calculate the amount of heat spent in internal work during the conversion of water at 100° C. into steam at the same temperature? 496.1 calories per gram.

33. The dew point of air at 25° C. is found to be 17° C. Assuming the vapor tensions of water at 17° C. to be 14.4 mm. of mercury, determine the relative humidity. What would be the relative humidity if the temperature of the air had been 20° C.?

61.2 per cent; 82.7 per cent.

34. How much heat is absorbed when 1 kg. of liquid air is boiled under atmospheric pressure and subsequently heated to 20° C.? Compare this amount with the heat absorbed by 1 kg. of ice melted at 0° C. and the water subsequently heated to 20° C. (Boiling-point of air $= -190°$ C.; heat of vaporization of air 50 calories per gram.)

99,770 calories; 100,000 calories.

35. The coefficient of conduction of copper is 0.96 calorie per cu. cm. How many heat units will pass per minute across a plate 1 m. long, 1 m. broad and 1 cm. thick when its opposite faces are kept at temperatures differing by 10°? 5.76×10^6 calories.

36. An iron boiler is 1 cm. thick and has a heating surface of 2 sq. m. The water in the boiler is at 100° C. and the heating surface is kept at 280° C. Taking the latent heat of steam as 536, and the conductivity of iron as 0.16, find how much water can be evaporated per hour? 3869×10^3 g.

37. A mirror is made to revolve about a vertical axis 25 times a second. If a horizontal beam of light be allowed to fall on the mirror

from a fixed source, required the velocity at which the reflected beam would traverse a circle 78 cm. in diameter having its center on the axis of the mirror. 1.23×10^4 cm. per second.

38. When a spherometer is placed on a concave mirror the movable leg has to be moved from a position in a plane with the fixed legs 0.316 cm. to come in contact with the face of the mirror. The distance between the fixed legs of the spherometer is 2 cm. What is the focal length of the mirror? 1.13 cm.

39. A concave mirror, radius of curvature 80 cm., forms an image of an object.

1. Object is 20 cm. in diameter. 200 cm. distant from mirror.
2. Object is 5 cm. in diameter. 50 cm. distant from mirror.
3. Object is 5 cm. in diameter. 30 cm. distant from mirror.

Find nature, position and size of image in each case.

50 cm. from mirror 5 cm. diam.; 200 cm. from mirror 20 cm. diam. 120 cm. from mirror 4 cm. diam.

40. When a layer of liquid 4.65 cm. deep is poured upon a dot on a glass plate, the position of the image as found by the necessary change of the microscope is 1.37 cm. above the plate. What is the index of refraction of the liquid? 1.42.

41. The minimum deviation produced by a hollow prism filled with a certain liquid is 53°. If the refracting angle is 60°, what is the index of refraction of the liquid? 1.668.

42. An equi-convex lens whose focal length is 80 cm. and index of refraction 1.6 is to be reground so that its focal length will be 100 cm. How much will the thickness of the lens be altered, the diameter remaining the same as before at 16.43 cm.? 0.14 cm.

43. 1. A lens is made of glass, of which the refractive index is 1.64. The radius of curvature of one surface of the lens is 20 cm. and the other surface of the lens is flat. What is the principal focal length of the lens? 2. What must be the refractive index for an equi-convex lens so that the focal length will be equal to the radii of curvature? 31.25; 1.5.

44. An object is placed 15 cm. from a diverging lens, of which the focal length is 20 cm. Find the nature and position of image, and ratio of diameter of object to diameter of image. 8.57; 1 : 0.571.

45. A projection lantern takes a slide 7×7 cm. The focal length of the lantern objective is 20 cm. It is desired to project on the screen an image of the slide 2 m. square. Required the distance from the lens to the screen. 5.91 m.

46. A compound microscope has an objective of which the focal length is 2 mm., an eyepiece of which the focal length is 10 mm., and the distance from the center of objective to the plane of image is 150 mm. Calculate the magnifying power of the instrument. 1924.

47. The object glass of a telescope at the Lick observatory is 92 cm. free diameter and has a focal length of 1500 cm. What is its numerical aperture and what is its magnifying power with an eyepiece of 2 cm. focal length? 0.0613; 750.

48. If the nearest distance for distant vision for a far-sighted person is 2 ft. 11 in., what should be the focal length of the spectacles he would require for reading? 14 in.

49. The true period of rotation of Jupiter's second satellite is eighty-five hours, and the velocity of the earth in its orbit is 30 kilometers per second. Find the apparent period of rotation of the satellite when the earth is moving directly away from Jupiter; when the earth is moving directly towards Jupiter.

85 hours, 0 minutes, 30.6 seconds;
84 hours, 59 minutes, 29.4 seconds.

50. In using Foucault's method for determining the velocity of light, it was observed that when the mirror was turning 257 times per second the displacement of the image was 0.113 m. The distance between the slit and the moving mirror was 8.58 m., and between the two mirrors 605 m. From these values determine the velocity of light. 2.96×10^{10} cm. per second.

51. A ray of light strikes a plane parallel plate of glass of index of refraction 1.5, at an angle of 70°, and emerges from the other side parallel to its original direction, but displaced laterally through a distance of 5 mm. How thick is the glass? 0.752 cm.

52. Light from two exactly similar and equal sources very close together falls on a screen. The distance between the sources of light is 0.184 cm. and the distance between the sources and the screen is 112 cm. A series of bright and dark bands at a distance of 0.036 cm. apart is observed on the screen. Find the wave length of the light used. 5914×10^{-8} cm.

53. A glass plate 0.00004 cm. thick is illuminated by a beam of light making an angle of 30° to the normal of the plate. The index of refraction of the glass is 1.5. What wave length of light will be strengthened by interference? 4525×10^{-8} cm.

54. The deviation of a line in the second spectrum of a diffraction grating was found to be 42° 8′. The grating contained 14,440 lines to the inch. What was the wave length of the light used?

5899×10^{-8} cm.

55. What is the angle of minimum deviation produced by a grating containing 10,000 lines to the inch, for the " C " line, wave length 6563×10^{-8} cm.? 14° 50′.

56. In a direct-vision spectroscope, the flint-glass prism has an angle of 54°. Compute the base angles of the two crown-glass prisms needed to give direct vision for the D line. 31° 42′.

57. The intensity of the light which is transmitted through a polariscope with parallel Nicols is taken as unity. What is the intensity of the light when the analyzer is turned through 15°, 60°, 90°? What is the angle where the intensity of the transmitted ray is $\frac{1}{2}$ that of the incident? 0.933; 0.250; 0; 45°.

58. The rotation produced by a 25 per cent solution of sugar in a 20 cm. tube is for sodium light 36°.33. The constant for sugar is 6.65. What is the percentage purity of the sugar? The density of a 25 per cent solution = 1.105. 99 per cent.

59. A crystal of Iceland spar cut with faces parallel to the axes is 2 cm. thick. How far below the upper surface are the ordinary and extraordinary images of a pencil mark on the lower face?
1.206; 1.345.

60. A standard candle and a gas flame are placed at a distance of 6 ft. apart. The gas flame is 4 candle power. Where must a screen be placed on a line joining the candle and the gas flame so that it may equally illuminated by both of them? 2 ft. in front or 6 ft. behind.

61. If the light of the full moon is found to produce the same illumination as a standard candle does at 4 ft., what is the equivalent in candle power of the moon's light? Take the distance of the moon as 240,000 miles. 10.04×10^{16} candle power.

62. A certain photographic lens gives a good photograph with an exposure of $\frac{1}{50}$ second when the sun is 75° above the horizon. What exposure would be required with the same lens when the sun is 5° above the horizon, ignoring increase of atmospheric absorption with increase of zenith distance of sun? 0.222 second.

63. On mapping the spectral intensity curve of an incandescent source it is found that the maximum intensity is at wave length 12,000 Angstrom units. What is the temperature of the source?
T = 2144° C.

64. The displacement of the F line of hydrogen (wave length 4861 Angstrom units) in the spectrum of a star is 0.1 of a unit toward the violet. What is the direction of motion and velocity of the star in the line of sight? 617,000 cm. per second.

PROBLEMS IN APPLIED MECHANICS USED AT MASSACHUSETTS INSTITUTE OF TECHNOLOGY *

1. Find the stresses in the members AB and AC of the frame shown (Fig. 1) by *three* different methods.

2. Find the stresses in the members AB and BC due to a vertical load of 3000 lb. acting on the frame as shown. (Fig. 2.)

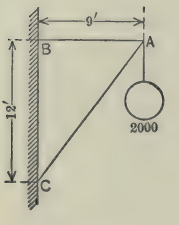

FIG. 1.

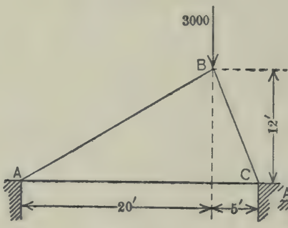

FIG. 2.

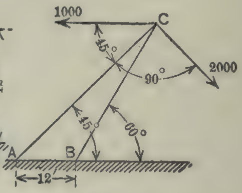

FIG. 3.

3. Find the stresses in the members AC and BC due to the forces acting at C as shown. (Fig. 3.)

4. Find the stresses in the members AB, BC, CD and BD of the derrick shown (Fig. 4) due to a load of 4000 lb. suspended at D.

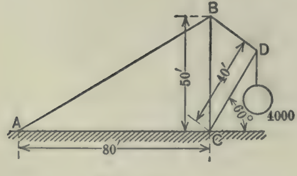

FIG. 4.

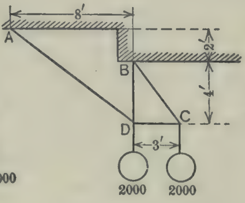

FIG. 5.

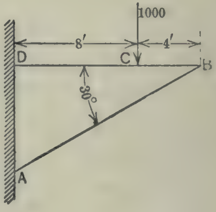

FIG. 6.

5. Find the H. and V. components of the supporting forces at the joints A and C of the frame given in Problem 4.

6. Find the stresses in the members CD, CB, BD and AD of the frame shown due to the two loads of 2000 lb. shown. (Fig. 5.)

7. Find the H. and V. components of the supporting forces at the points A and B of the frame given in Problem 6.

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* Used by permission of Profs. Fuller and Johnson.

8. Find the stresses in AB and the H. and V. components of the forces acting at D on the frame shown. (Fig. 6.)

9. Find the stress in BD and the force acting at A in the frame shown. (Fig. 7.) Find the angle that this force makes with the horizontal.

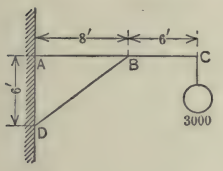


FIG. 7.

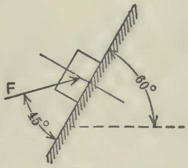


FIG. 8.

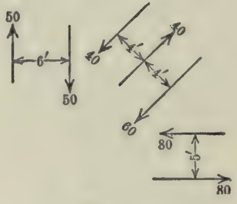


FIG. 9.

10. Find the force F, necessary to keep the weight of 200 lb. from sliding down the frictionless plane shown. (Fig. 8.)

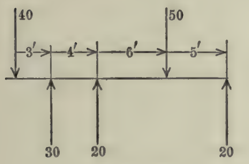


FIG. 10.

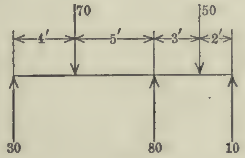


FIG. 11.

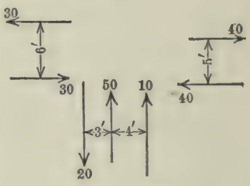


FIG. 12.

11. Solve Problem 11 on p. 29 in text-book, using pressure P equal to 1500 lbs. (Fuller and Johnson.) (Omit.)

12. Find the resultant of the system of force shown. (Fig. 9.)

13. Determine the resultant of the system of parallel forces shown. (Fig. 10.)

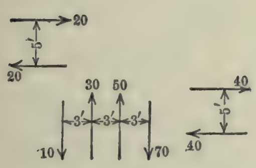


FIG. 13.

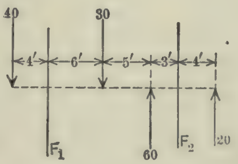


FIG. 14.

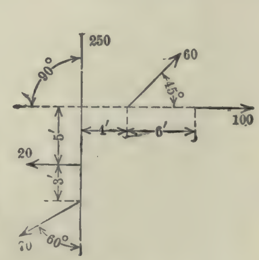


FIG. 15.

14. Determine the resultant of the system of parallel forces shown. (Fig. 11.)

15. Determine the resultant of the system of parallel forces and couples shown. (Fig. 12.)

16. Determine the resultant of the system of parallel forces and couples shown. (Fig. 13.)

17. The system of parallel forces shown is in equilibrium. Find the *magnitudes* and *directions* of the unknown forces F_1 and F_2. (Fig. 14.)

18. Find the resultant of the system of forces shown (Fig. 15) and indicate its position on sketch.

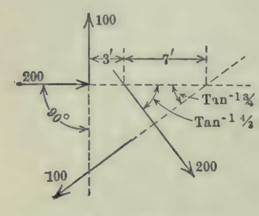


FIG. 16.

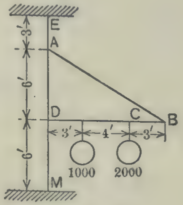


FIG. 17.

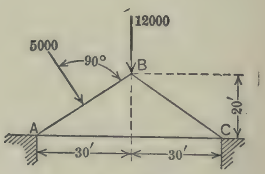


FIG. 18.

19. Find the resultant of the system of force shown (Fig. 16) and indicate its position on sketch.

20. Assuming that the reaction at E is horizontal, find its magnitude and the H. and V. components of the reaction at M. (Fig. 17.) Also find the stress in AB and the H. and V. components of the force at D.

21. The frame ABC (Fig. 18) is loaded with a vertical load of 12,000 lb. at B and a load of 5000 lb. perpendicular to the member AB at its middle point. Assuming that the supporting force at A is vertical, find its magnitude and the magnitude of the H. and V. components of the supporting force at C.

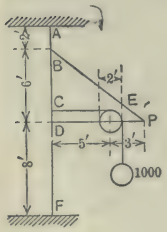


FIG. 19.

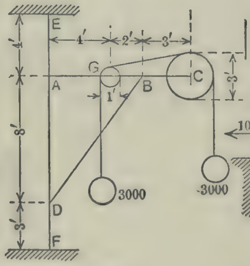


FIG. 20.

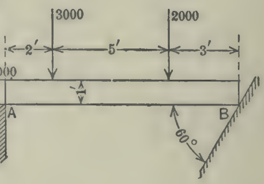


FIG. 21.

22. Determine the stresses in the members AC and BC of the frame shown in Problem 21.

23. Assuming that the reaction at A (Fig. 19) is horizontal, find its magnitude and the H. and V. components of the reaction

at *F*. Also find stress in *BP* and H. and V. components of the force acting at *D* on *DP*.

24. Find the H. and V. components of the supporting forces at *E* and *F* and the H. and V. components of the forces acting at the joints *A*, *B*, and *D*. (Fig. 20.) Assume reaction at *E* to be horizontal.

25. Find the H. and V. components of the supporting forces at *A* and *B* due to the forces of 3000 lb., 2000 lb., and 1000 lb. shown, assuming that the member is hinged at *A* and that there is no friction at *B*. (Fig. 21.)

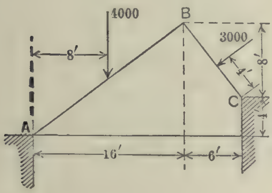


FIG. 22.

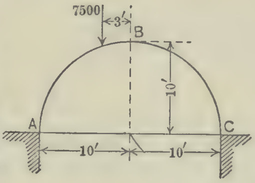


FIG. 23.

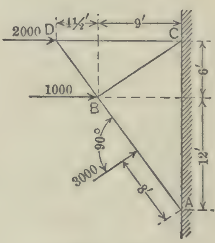


FIG. 24.

26. Find H. and V. components of the forces acting at joints *A*, *B*, and *C* of frame. (Fig. 22.)

27. The semicircular arch is made up of two arched ribs *AB* and *BC*, hinged together at *B* and to the supports at *A* and *C*. Find the H. and V. component reactions of each of the three hinges due to a load of 7500 lb. acting as shown. (Fig. 23.)

28. Find the stress in the member *BC* and the H. and V. components of the forces acting at *A* and *C* in the frame shown. (Fig. 24.)

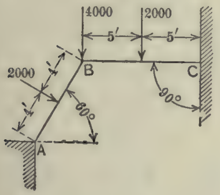


FIG. 25.

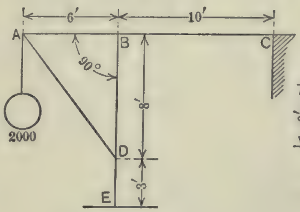


FIG. 26.

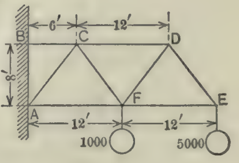


FIG. 27.

29. Find H. and V. components of forces acting at joints *A*, *B*, and *C* of frame. (Fig. 25.)

30. The member *AC* is horizontal and is supported at *B* and *C*. If the rope supporting weight is fastened to a cleat at *D*, find the H. and V. components of the forces at *C* and *E* and the H. and V. reactions of *BE* on *AC* at *B*. Neglect diameter of pulley. (Fig. 26.)

31. The truss shown (Fig. 27) is held by fixed pins at A and B and supports the loads shown. Find the H. and V. components of the supporting forces at A and B and the stresses in all the members.

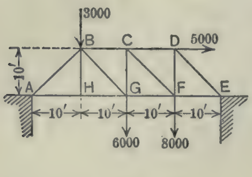


FIG. 28.

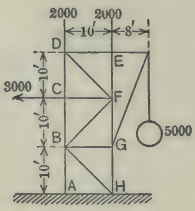


FIG. 29.

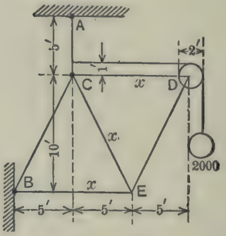


FIG. 30.

32. Find the H. and V. components of the supporting forces at A and E (Fig. 28). Assume reaction at A vertical. Find the stresses in all the members.

33. Find the H. and V. components of the supporting forces at A and H and the stresses in all the members of the truss shown. (Fig. 29.)

34. Find the H. and V. components of the supporting forces at A and B and the stresses in all the members marked X. (Fig. 30.)

35. Solve Problem 5 on page 57 in text-book replacing the force of 8000 lb. with one of 5000 lb. (Fuller and Johnson.) (Omit.)

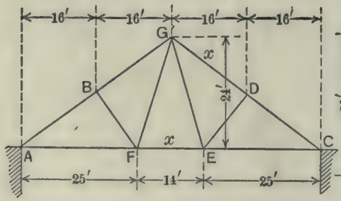


FIG. 31.

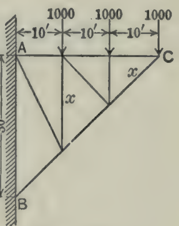


FIG. 32.

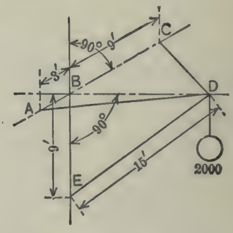


FIG. 33.

36. Find the stresses in the members marked (X) in the roof truss shown (Fig. 31), loaded with a dead load of 16,000 lb., and a wind load of 8000 lb. on the *right*-hand side. The truss is fixed in position at (C) and supported on a roller nest at (A).

37. Find the H. and V. components of the supporting forces at A and B and the stresses in the members marked X of the frame shown. (Fig. 32.)

38. Find the stresses in the members AD, CD and DE due to the load of 2000 lb. shown. (Fig. 33.)

39. Find the stresses in the members *CD* and *AD* and the H. and V. components of the supoprting force at *E*, due to the two loads shown. (Fig. 34.)

40. In the frame shown (Fig. 35) the vertical plane through *ABC* intersects the horizontal plane through *C* in the line *CF*. Find the stresses in all the members of the frame due to the load of 6000 lb. at *A*.

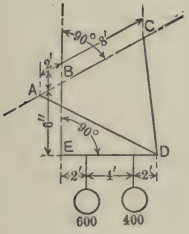


FIG. 34.

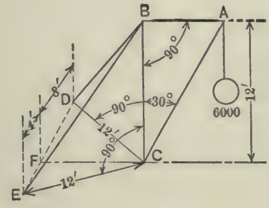


FIG. 35.

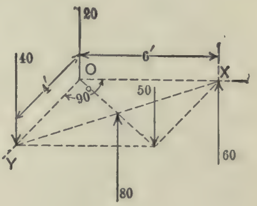


FIG. 36.

41. Find the resultant of the system of parallel forces shown. (Fig. 36.)

42. Find the resultant of the system of parallel forces shown in Problem 41, if the force of 60 lb. is replaced by a force of 30 lb. acting in the same direction.

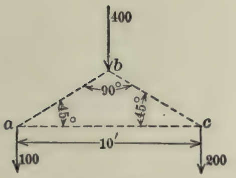


FIG. 37.

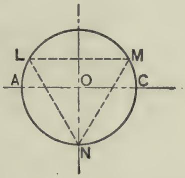


FIG. 38.

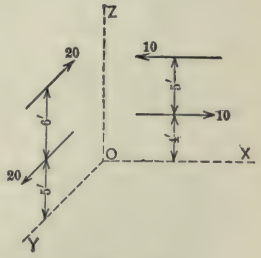


FIG. 39.

43. The forces shown are perpendicular to the plane of the triangle *abc*. (Fig. 37.) Find the force which will *balance* the system and indicate position by sketch

44. A circular table (Fig. 38) 9 ft. in diameter, is supported on three legs, *L*, *M*, and *N*, at equal distances apart. If a weight of 40 lb. is placed at *A*, 30 lb. at *O*, and 80 lb. at *C*, find the stress in each leg. The diameter *AC* is parallel to *LM*.

45. Find the resultant of the couples located in the *X* and *Y* planes as shown. (Fig. 39.)

46. Find the resultant of the system of parallel forces shown. (Fig. 40.) Plot it.

47. The couples shown (Fig. 41) are parallel to the X, Y, and Z planes respectively. Find the resultant couple and show its position on sketch.

48. The forces 10 lb. and 20 lb. (Fig. 42) are in the Y plane and parallel to OZ; the forces 60 lb. and 40 lb. are in the Z plane and parallel to OY; the forces 30 lb. and 50 lb. are in the X plane and parallel to OZ. Find the resultant of the system and indicate its position by sketch.

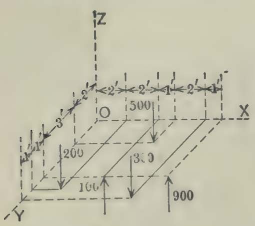


FIG. 40.

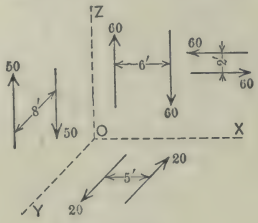


FIG. 41.

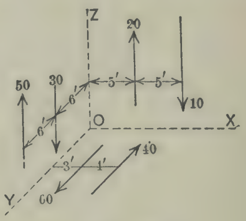


FIG. 42.

49. Find the resultants of the distributed forces shown (Fig. 43) and locate their lines of actions in sketch.

Case A. Intensity at (a) and (b) equals zero and 100 lb. per foot respectively.

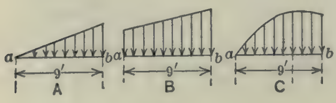


FIG. 43.

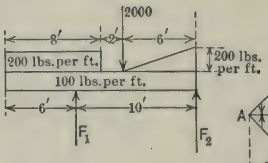


FIG. 44.

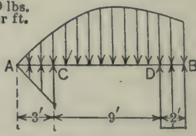


FIG. 45.

Case B. Intensity at (a) and (b) equals 150 and 300 lb. per foot respectively.

Case C. Intensity at (a) equals zero and at distance x from (a) equals $12x - x^2$ lb. per foot respectively.

50. The beam shown (Fig. 44) is subjected to the distributed forces and concentrated load of 2000 lb. as indicated. Find the supporting forces F_1 and F_2.

51. A force is distributed along a line AB (Fig. 45) in such a manner that its intensity at any point at a distance x from A, is equal to $p = 10x - \frac{x^2}{4}$, and is balanced by a uniformly varying force

acting along AC, and a uniform force acting along DB. Find the intensities of the balancing forces at the points A, C, D, and B.

52. Locate the center of gravity of the cross-section of the angle shown. (Fig. 46.)

53. Locate the center of gravity of the cross-section of the channel shown. (Fig. 47.)

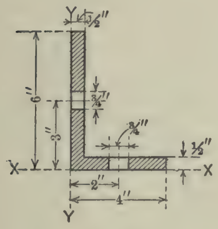


FIG. 46.

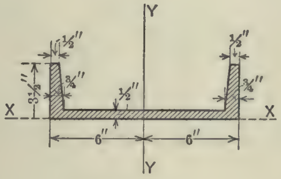


FIG. 47.

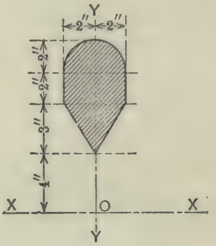


FIG. 48.

54. Locate the center of gravity of the cross-section shown. (Fig. 48.)

55. Locate the center of gravity of the cross-section of the T beam (Fig. 49) made by riveting together two plates and two angles.

56. Locate the center of gravity of the shaded area with respect to the axes XX and YY. (Fig. 50.)

57. Locate the center of gravity of the outer perimeter of the area given in Problem 56.

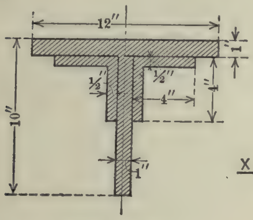


FIG. 49.

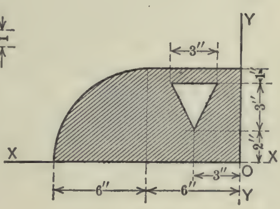


FIG. 50.

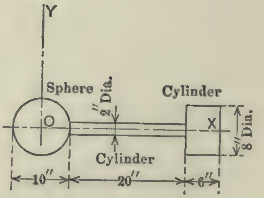


FIG. 51.

58. Determine the volume generated by revolving the area given in Problem 56 about the axis XX.

59. (*a*) Find the center of gravity of the solid shown. (Fig. 51.) (*b*) How could the center of gravity of this body be found by experiment? (Indicate a method other than by balancing.)

60. Find the moments of inertia of the area shown in Problem 56 about the axes XX and YY. Also find moments of inertia about axes passing through center of gravity parallel to XX and YY.

Express results in both inch-units and foot-units. Determine also the polar moment of inertia of the area with respect to an axis through *O*. (Fig. 50.)

61. Find the moment of inertia of the cross-section of a hollow circular cylinder about its diameter; outside diameter of cylinder = 12 in. and inside diameter = 10 in. Find the polar moment of inertia of the section with respect to an axis through its center.

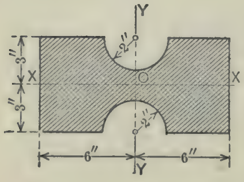


FIG. 52.

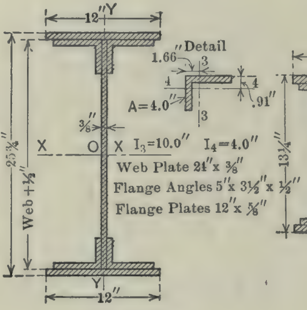


FIG. 53.

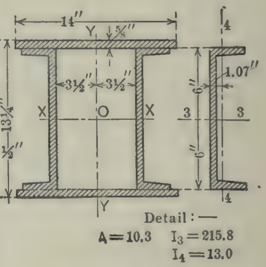


FIG. 54.

62. Find the moments of inertia of the shaded area with respect to the axes *XX* and *YY*. (Fig. 52.)

63. Find the moment of inertia of the section of the girder shown (Fig. 53) about axis *XX* in both inch- and foot-units.

64. Find the moment of inertia about the axes *XX* and *YY* passing through the center of gravity of the cross-section of the column shown. (Fig. 54.) Also find radius of gyration about each axis.

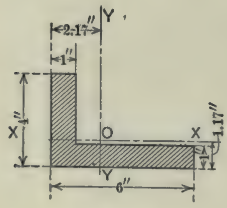


FIG. 55.

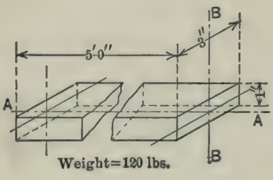


FIG. 56.

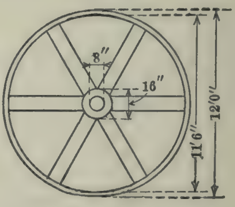


FIG. 57.

65. Determine the moments of inertia of the area shown in Problem 53 with respect to the axes *XX* and *YY*.

66. Determine the moment of inertia of the area shown in Problem 54 with respect to the axis *XX*.

67. The axes *XX* and *YY* pass through the center of gravity

of the shaded area (Fig. 55). If $I_{xx}=4.3$ in. units, $I_{yy}=16.6$ in. units and the Area$=4.50$ sq. in., determine the least radius of gyration for the section.

68. Find the moment of inertia of a homogeneous rod 5 ft. long, of rectangular cross-section (1×3 in.) about the axis *AA* passing through the center of gravity of the body (Fig. 56). Also find the moment of inertia about the axis *BB* perpendicular to *AA* by the exact and approximate methods.

69. Find the moment of inertia of the flywheel about its axis. (Fig. 57.) Width of hub$=16$ in. Width of rim$=26$ in. Area of cross-section of each arm$=25$ sq. in. Wt. of iron$=450$ lb. per cubic foot. Give results in both inch-pound units and foot-pound units.

70. A weight of 3 tons, starting from rest, is raised 100 ft. in four seconds by a constant pull on the hoisting rope. Find the tension in the rope. If, at the end of four seconds, the tension in the rope is reduced to 1 ton, how much higher will the weight rise?

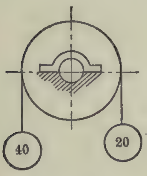


FIG. 58.

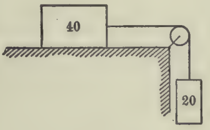


FIG. 59.

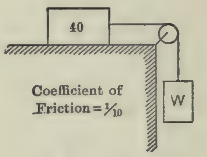


FIG. 60.

71. Find the acceleration of the weights and the tension in the cord in each case (Figs. 58 and 59). Neglect friction and weight of pulley and cord. In the case (Fig. 59) also find the tension in the cord, assuming the coefficient of friction between the weight and the plane to be $\frac{1}{10}$.

72. The weight shown (Fig. 60) starts from rest and move through a distance of 100 ft. At the instant the weights pass the 100-ft. mark the cord is cut and one second later the velocity of the weight *W* is 80 ft. per second. Find the magnitude of the weight *W*, neglecting the weight of the pulley and cord.

73. A weight of 40 lb. starting from rest is moved along a horizontal plane by a force which is equal to $10+\frac{t^2}{12}$.

If the coefficient of friction between the weight and the plane is constant and equal to 0.1, find the space passed over in ten seconds and the velocity at the end of that time. If the force ceases to act at that time how much farther would the weight move?

74. A weight of 100 lb. is moved along a horizontal plane by a force which varies in such a manner that $s=\frac{t^4}{20}$.

If the friction between the weight and the plane is constant and equal to 15 lb., find the magnitude of the force at the end of ten seconds.

75. A weight of 200 lb., starting from rest, is moved along the inclined plane by the forces shown. (Fig. 61.) If the coefficient between the weight and the plane is constant and equal $\frac{1}{10}$, find the distance the weight will move in one-half minute and the velocity at the end of that time.

76. Find the acceleration of the weights and tension in the cord, neglecting the weight of the cord and pulley (Fig. 62). The coefficient of friction between the weights and planes is $\frac{1}{10}$. If the weights start from rest, find the change in kinetic energy in five seconds. Find the minimum horizontal force applied to the 60 lb. weight which will be required to produce equilibrium.

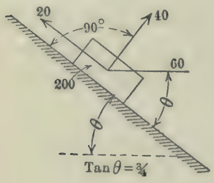


FIG. 61.

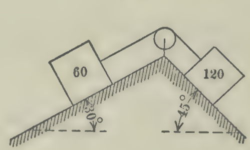


FIG. 62.

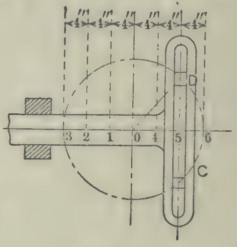


FIG. 63.

77. The speedometer of a car running on a straight and level road gave the following readings:

Time	0	10	20	30	40	50	60	70	80	Secs.
Speed,	0	4.2	7.1	9.5	11.1	12.1	12.4	12.5	12.5	miles hr.

Plot the velocity-time curve and the space-time curve.

Use for scales
Vel.—Ord. 1 sec. = 2 m.p.
Time—Abs. 1 sec. = 10 secs.
Space—Ord. 1 sec. = 200 ft.

Find the acceleration of the car at the end of thirty secs. in $\frac{\text{ft.}}{\text{sec.}^2}$. Where is the acceleration the greatest and where the least? Where is the unbalanced force acting on the car the greatest and where the least?

78. A steam hammer with piston weighs 2 tons. If the area of the piston is 100 sq. in., and the average steam pressure is 100 lb. per square inch, and the stroke is 3 ft., what is the maximum available energy of the blow? Neglect friction. Cylinder vertical.

79. A train weighing 500 tons is drawn up a grade of 50 ft. per mile by an engine exerting a constant tractive force of 20,000 lb. If the grade is 1 mile long and the velocity at the bottom is 2 miles per hour, find the velocity at the top of the grade. Find the horse-power the engine exerts at the bottom of the grade and also at the top. Find the horse-power necessary to keep the train moving up the grade at a uniform speed of 30 miles per hour. Assume the train resistance to be constant and equal to 12 lb. per ton.

80. If the crank-pin B (Fig. 63) rotates at a speed of 160 R.P.M. and the weight of the cross-head C, moving in harmonic motion, is 200 lb., find the magnitude and *direction* of the *unbalanced* force acting on the cross-head C when in each of the seven positions indicated. Determine also the velocity of C when in each of the seven positions.

81. A weight of 100 lb. starting from rest is raised by a force which decreases as the body rises. The initial magnitude of the force is 240 lb. acting upward and it decreases directly as the distance through which the body rises. It is 100 lb. acting upward when the body has risen 100 ft. What is the velocity at the 100-ft. mark? What is the acceleration at this time? If the force continues to decrease at the same rate *until the body stops*, what is the maximum height attained?

82. A weight of 20 lb. falls from a height of 14 in. and strikes a flat spring, deflecting it 4 in. The deflection of the spring varies directly as the force applied. Find the maximum pressure exerted by the spring.

83. A weight of 60 lb. moving horizontally with a velocity of 10 ft. per second strikes a helical spring whose resistance to compression is proportional to the forces applied, viz.: 200 lb. for 1 in., 400 lb. for 2 in., etc. Find maximum amount spring is compressed.

84. A weight of 300 lb. is moved from A to B (Fig. 64) by a varying force whose magnitude at any point is indicated by the ordinate of the diagram. The coefficient of friction between the weight and the plane is equal to $\frac{1}{20}$. If the velocity at A is 10 ft. per second, find the velocity at B. Find the H.P. exerted when the weight is at C; when it is at B.

85. A straight rod 2×2 in. cross-section and 8 ft. long rotates about an axis passing through the rod perpendicular to its length

2 feet from one end at a speed of 200 R.P.M. If the weight of the material is $\frac{1}{4}$ lb. per cubic inch find the pull on the axis and the maximum tension due to centrifugal force. Find the tension in pounds per square inch at a cross-section 2 ft. from the axis; at 4 ft. from axis; at 6 ft. from axis. Prove that the intensity of the tension is independent of the cross-section of the rod.

86. The two spheres connected by a rod 2 in. diameter (Fig. 65) revolve about an axis perpendicular to the rod through O at a speed of 500 R.P.M. Find the pressure on the axis due to centrifugal force. Weight of material equals $\frac{1}{4}$ lb. per cubic inch.

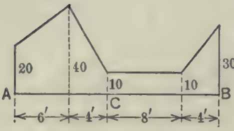


FIG. 64.

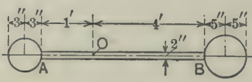


FIG. 65.

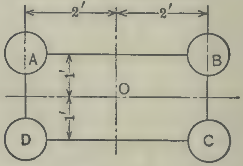


FIG. 66.

87. Four balls (Fig. 66) each weighing 100 lbs., rotate about an axis through O perpendicular to the plane of the paper at a speed of 200 R.P.M. Find the tension in the rods, *connecting the weights*, neglecting the effect of gravity and the weight of the rods.

88. A thin ring 5 ft. in diameter (weighing 100 lb.) revolves about an axis through its center, perpendicular to its plane, at a speed of 200 R.P.M. Find the tension in the ring.

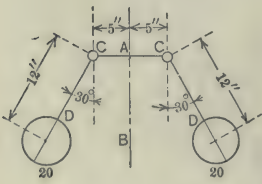


FIG. 67.

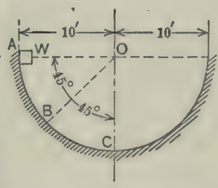


FIG. 68.

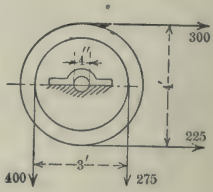


FIG. 69.

89. The weight of 20 lb. (Fig. 67) revolves about the axis AB as shown. Find speed in revolutions per minute when the rods CD make an angle of 30° with the vertical. Also find tension in rods CD. Neglect weight of rods.

90. Find the tension due to centrifugal force in a belt weighing 4 lb. per foot of length traveling at a speed of 4000 ft. per minute.

91. A weight W starting at the point A (Fig. 68) slides down

the quadrant of a circle of 10 ft. radius. Find the pressure between the weight and the curve when the weight reaches the point B and also when it reaches point C. Neglect friction.

92. Given a solid cylinder 3 ft. diameter, weighing 500 lbs. mounted on a shaft, revolving at a speed of 100 revolutions per minute. If the friction at the bearings of the shaft is 36 in.-lbs., find the resultant force (F) acting tangent to the rim which will increase the speed to 300 R.P.M. in thirty seconds. Find increase in kinetic energy.

93. If the cylinder in Problem 92 is revolving at 300 R.P.M., and the force F is removed, find the time in which the cylinder will come to rest. Also find the number of revolutions it will make before stopping.

94. The pulleys shown (Fig. 69) are mounted on a shaft 4 in. in diameter and weigh 600 lb. Moment of inertia of pulleys = 2000 ft.-lb. units. If the shaft starts from rests find speed in revolutions per minute at the end of one minute. Find resultant pressure on shaft.

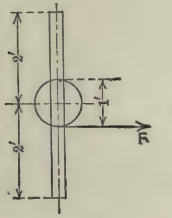


FIG. 70.

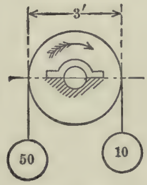


FIG. 71.

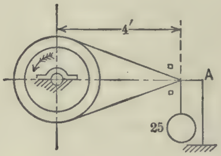


FIG. 72.

95. If the pulleys in Problem 94 revolve at a uniform speed of 250 R.P.M. under the action of the forces shown and friction, find the friction loss in horse-power.

96. If a flywheel whose moment of inertia is equal to 1,200,000 ft.-lb. units revolves at a speed of 100 R.P.M., find the work that can be done by the stored energy without reducing the speed more than 2 per cent.

97. A rod of uniform section and 4 ft. long weighing 60 lb. is fastened to the end of a thin hollow cylinder weighing 40 lb. as shown. (Fig. 70.) If the initial velocity is 100 R.P.M., find the velocity at the end of ten seconds, if the force F is constant and equal to 20 lb.

98. The pulley shown (Fig. 71) rotates at a speed of 500 R.P.M. under a band which is held in place by the weights shown. Find

the horse-power absorbed by the friction between the band and the pulley.

99. The pulley shown (Fig. 72) rotates at a uniform speed of 800 R.P.M. under a band which is held in place by the frame, and the friction between the band and the pulley is just sufficient to balance the outer end as shown at *A*. If the weight of the frame is 100 lb. (exclusive of the 25 lb. shown) and its center of gravity is 1 ft. to the right of the center of the pulley, find the horse-power absorbed by the friction.

100. Find acceleration of weights and maximum tension in cord, taking into account the weight of disk which is 5 lb. Neglect weight of cord. (Fig. 73.)

101. A solid cylinder weighing 200 lb. starting from rest is rolled along a horizontal plane by constant force of 20 lb. (Fig. 74.) Find the friction between the cylinder and the plane. (Neglect rolling friction.) How far will the weight roll in six seconds. Find linear velocity at the end of that time.

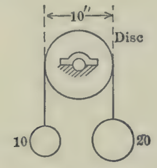


FIG. 73.

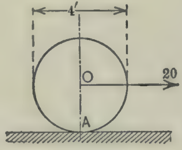


FIG. 74.

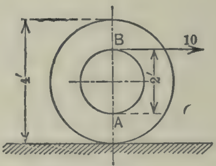


FIG. 75.

102. If the sliding friction between the cylinder and plane given in Problem 101 is 4 lb., how far will the weight move in ten seconds?

103. In Problem 102 locate the instantaneous axis and determine the kinetic energy at the end of ten seconds.

104. Given a solid cylinder weighing 200 lb. (Fig. 75.) If the cylinder starts with an initial velocty of 5 ft. per second, how far will it move in four seconds under the action of the horizontal force of 10 lb. acting at *B* as shown, assuming no slipping.

105. If the force acting on the cylinder given in Problem 104 is applied at *A* instead of *B* how far will the cylinder move in four seconds assuming no slipping?

106. A wheel (Fig. 76) weighing 100 lb.; moment of inertia about its axis = 600 lb.-ft. units rolls along a horizontal plane under the action of forces shown.

(*a*) If the coefficient of friction between the weight and the plane is equal to $\frac{1}{10}$, find the distance the wheel will roll in five seconds, if wheel starts from rest.

(*b*) If the coefficient of friction $= \frac{1}{50}$, find the distance the wheel will roll in five seconds, starting from rest.

(*c*) In each case determine the kinetic energy at the end of five seconds.

107. Given a solid sphere (Fig. 77) weighing 30 lb. rolling down the inclined plane. If the initial velocity is zero, find the coefficient of friction necessary to prevent slipping.

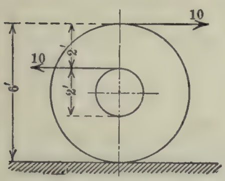


FIG. 76.

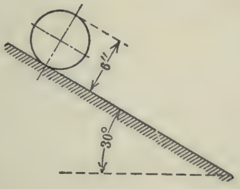


FIG. 77.

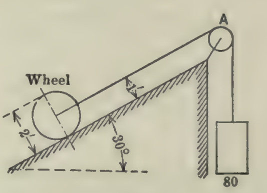


FIG. 78.

108. If the weight of the wheel (Fig. 78) equals 100 lb. and its moment of inertia equals 80 ft.-lb. units, finds its linear velocity after ten seconds, assuming initial velocity to be zero. Also find tension in the cord. Assume no slipping and neglect weight of pulley and cord.

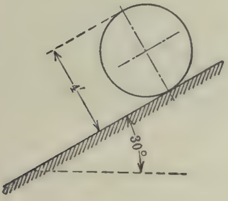


FIG. 79.

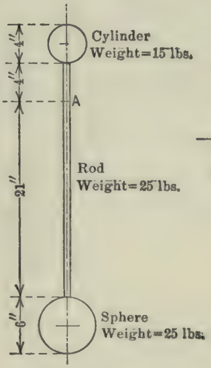


FIG. 80.

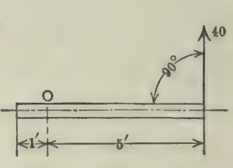


FIG. 81.

109. Solve Problem 108, taking into account the inertia of the wheel *A*. Assume diameter = 20 in.; $I_0 = 400$ in.-lb. units.

110. Given a wheel, weight 200 lb., starting from rest and rolling down an inclined plane. (Fig. 79.) If the moment of inertia of the wheel about an axis through the center of gravity = 500 lb.-ft. units, and the coefficient of friction = *W*, find:

(*a*) The linear acceleration of the center of the wheel;

(*b*) The friction at the surface;

(*c*) Energy of the wheel after ten seconds.

111. The weight shown (Fig. 80) is suspended on a horizontal axis through *A*. Find its center of percussion. Also find time of single oscillation when swinging as a pendulum.

112. A homogeneous rod of uniform section, weighing 60 lb., is acted upon by the force of 40 lb. as shown (Fig. 81). Find resultant pressure on the axis (*O*).

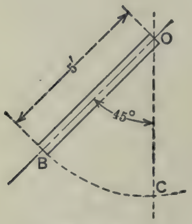


Fig. 82.

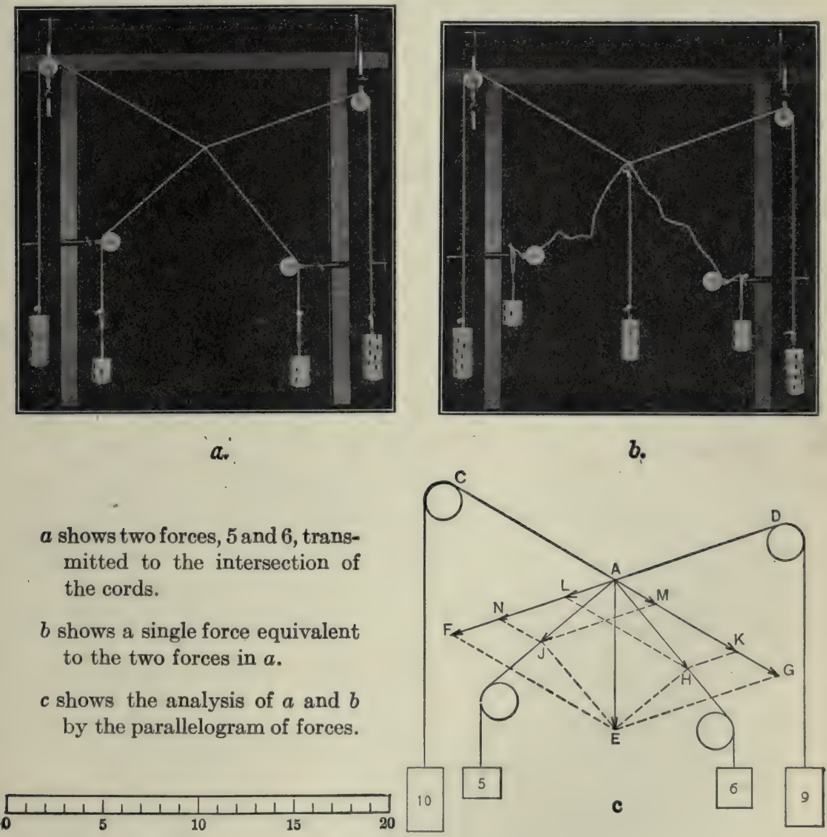
If the rod were in a vertical instead of a horizontal position what would be the resultant pressure on the axis?

113. A rod of uniform section suspended from a horizontal axis through *O* (Fig. 82) starting from rest at the point shown, swings downward under the action of gravity. Find the angular velocity when it reaches the vertical position *OC*.

If weight of rod is 30 lb., find pull on axis when it reaches position *OC*. Find resultant pull on axis at the starting point.

PROBLEMS IN MECHANICS AND HYDRAULICS FROM COURSE IN NATURAL AND EXPERIMENTAL PHILOSOPHY, U. S. M. A.*

Ex. 1. Lay off $AH = 6$ units of the scale and $AJ = 5$ units. The components of AH in the direction of AC and AD are AK and AL, and the components of AJ are AM and AN. Hence the stress in AC is equal to $AM + AK = AG$, and the stress in AD is $AL + AN = AF$.



a shows two forces, 5 and 6, transmitted to the intersection of the cords.

b shows a single force equivalent to the two forces in *a*.

c shows the analysis of *a* and *b* by the parallelogram of forces.

FIG. 1.

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Construct a parallelogram on AG and AF. The diagonal AE is also a diagonal of the parallelogram on AH and AJ, and a single force AE would have the same effect on the cords AC and AD as the two given forces.

The force AE is called the *resultant* of AH and AJ; and generally, *the resultant of a system of forces, is in general, a single force which would produce the same effect as the forces of the system acting together.*

The term *resultant* is also conveniently applied to a single force in relation to its components. For example, AE in Fig. 2 may be called the resultant of its components AG and AF. So also AH and AJ in Fig. 1 are called components of their resultant AE.

The principle of the parallelogram of forces is not limited to the directions of given supports, as in Figs. 1 and 2, but may be applied in finding the components of any force in any two directions. Thus the horizontal and vertical components of AC or AD may be found by a rectangular parallelogram.

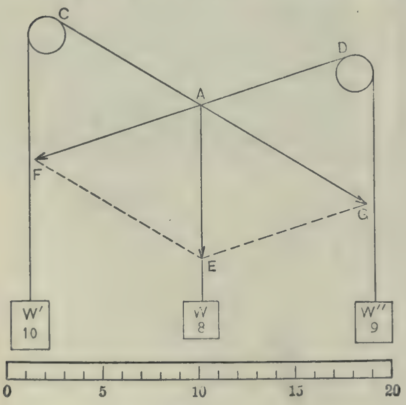

FIG. 2.

Ex. 2. (*a*) Construct the horizontal and vertical components of AG and AF in Fig. 2. Then combine the horizontal and vertical components separately, and show that the resultant of the four components is AE.

(*b*) Show the equilibrium of the point A in Fig. 2 by reversing AG and AF.

Ex. 3. Represent the equilibrium in Fig. 1 by indicating the stresses on the four cords, with their resultants.

Ex. 4. Fig. 3 shows a force, BD, applied vertically downward at the vertex B of the frame ABC, whose points of support are on abutments at A and C.

The stresses (compressions) in BA and BC are obtained by the parallelogram constructed on BD as a diagonal. The component of BD on BA is transmitted to A. Lay off AH equal to this component. The components of AH are AG and AI, giving the stress (tension) in AC and the direct pressure on the abutment. Similarly CL is decomposed at C, giving the tension CM and the direct pressure CK. The reactions at A and C are AI' and CK'.

The equilibrium is tested by $AG+CM=0$ and $BD+AI'+CK'=0$. That is, the algebraic sums of the horizontal and of the vertical components, including the reactions, are separately equal to zero.

Ex. 5. Solve the problem of Fig. 3 after substituting BD' for BD, assuming that A is fixed to the abutment while C is not.

Test the equilibrium, as in Ex. 4, by the zero value of the algebraic sums of the horizontal and of the vertical components.

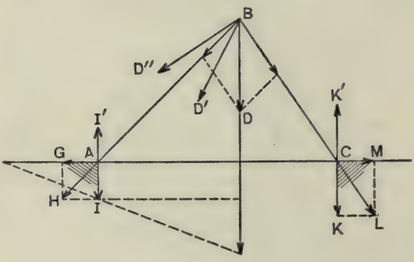

FIG. 3.

Ex. 6. Solve the problem of Fig. 3 after substituting BD' for BD, assuming that C is fixed to the abutment while A is not.

Test the equilibrium as in Ex. 2.

Ex. 7. Solve the problem of Fig. 3, Ex. 4, after substituting BD' for BD, assuming that C is fixed to the abutment while A is not.

Find the reaction of the abutments, and show that the resultant of these reactions is equal and directly opposed to the given applied force.

Ex 8. Fig. 4 shows a crane or derrick, with the following designated parts: mast, BD; tie, DE; boom, EB; guy, AD.

A given load at the vertex E is represented by EF. Find the stresses on the designed parts of the derrick, and the vertical and horizontal components of the reactions at A and B. Also show that the resultant of the reactions is equal and directly opposed to the load EF.

The problem is solved in the figure, as follows:

Stress (tension) on the tie $=Eg$.

Stress (compression) on the boom $=Eh$.

Eg, transmitted to D, produces tension on the guy, Dv'', and compression on the mast, Da'.

Eh, transmitted to B, produces the thrust Bh', whose horizontal and vertical components are Bd and Ba''.

Dv'', transmitted to A, produces the pull Av', whose horizontal and vertical components are Aw' and At.

The reaction at A is Av. The reaction at B is Bb', equal and opposed to the resultant of $Ba=Da'+Ba''$ and Bd. The resultant of the two reactions Av and Bb', transferred to e, is ef, equal and directly opposed to the load EF.

For the horizontal components, $Aw+Bd'=0$.

For the vertical components, $Ba'+(At'+EF)=0$.

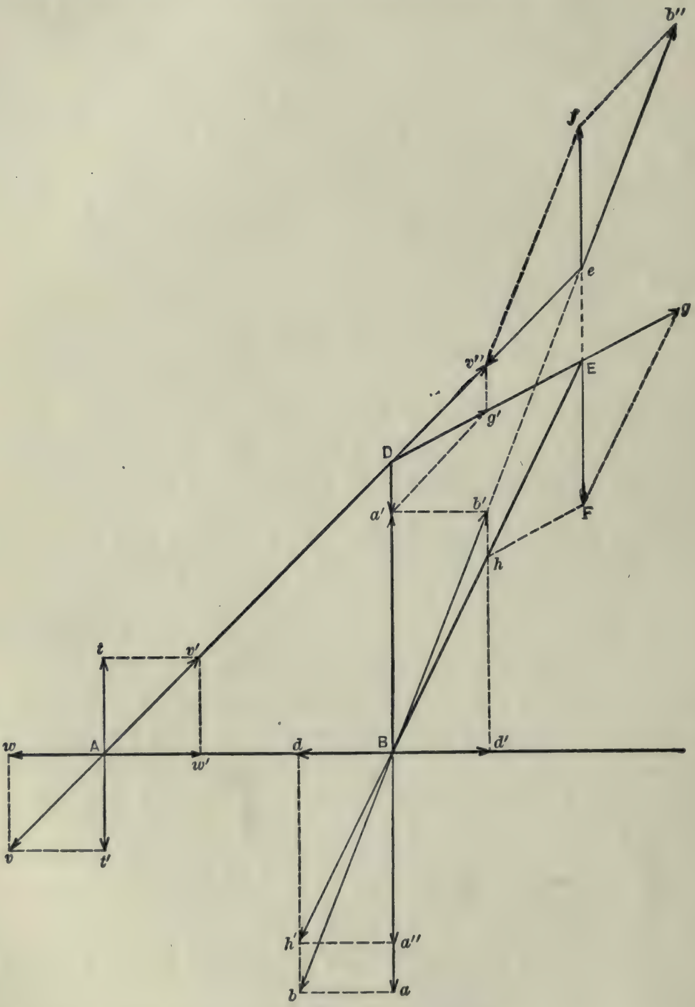

Fig. 4.

Ex. 9. In Fig. 5 introduce two forces, one at A and one at C, which shall produce equilibrium with BF and DG. (The action line of one of the forces may be assumed, after which the problem is determined.)

(*a*) **Two unknown forces of any coplanar system in equilibrium may be found when all the others are known, the action lines of the unknown forces being given.** Fig. 7 is an example, assuming that the three given forces in Fig. 6 are held in equilibrium by two other forces whose action lines are the given full lines through *d*. The resultant of the known forces may be found as already explained, *and it is a test of the equilibrium that the action line of this resultant must pass through the intersection of the action lines of the two unknown forces.* From this intersection lay off the intensity of the resultant of the known forces, *reversed in direction.* The components of this reversed resultant in the two known directions are the required intensities. In Fig. 7 the reversed resultant is *de*, and the required forces in the given directions through *d* are *df* and *dg*.

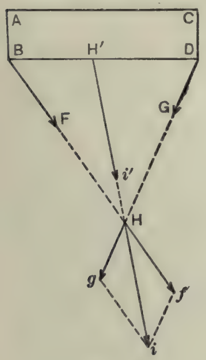

FIG. 5.

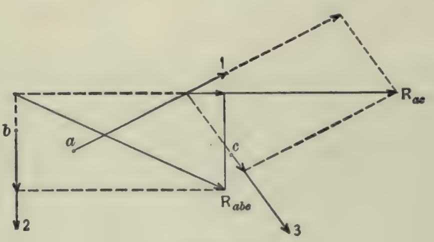

FIG. 6.

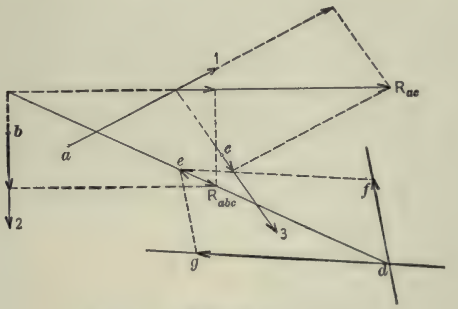

FIG. 7.

(The two known directions may be parallel. This case will be considered under *parallel forces.*)

(*b*) **Three non-concurrent unknown forces of a coplanar system in equilibrium may be found when all of the others are known, the action lines of the unknown forces being given.** In Fig. 8 let R' be the resultant of the known forces of the system, and let 1, 2, and 3 be the action lines of the unknown forces.

Prolong one of the given action lines, as 3, until it meets R' at A, and draw a right line from A to the intersection of 1 and 2 at B. The resultant of R' and 3 must lie on the line AB and be equal and directly opposed to the resultant of 1 and 2. The parallelogram on R' and 3, with diagonal on AB, gives F_3 and the resultant of R' and F_3. From B lay off the intensity of this resultant, *reversed in direction*, BR'''. The components of BR''' in the direction of 1 and 2 complete the solution, and the three required intensities are BF_1, BF_2, and AF_3.

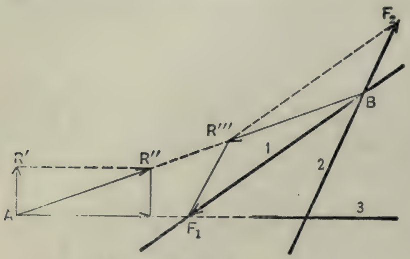

FIG. 8.

This construction fails when the unknown forces are concurrent or parallel.

Ex. 10. Scale Fig. 8 (scale per Fig. 1) and compute F_3 by moments with respect to B. Compare with the graphical solution.

(F_1 and F_2 have zero moments with respect to B, and therefore R' and F_3 must have equal and opposite moments about B in order to fulfill the conditions of equilibrium.)

Ex. 11. Scale Fig. 8 and compute F_1 by moments. Compare with the graphical solution.

Ex. 12. Scale Fig. 8 and compute F_2 by moments. Compare with the graphical solution.

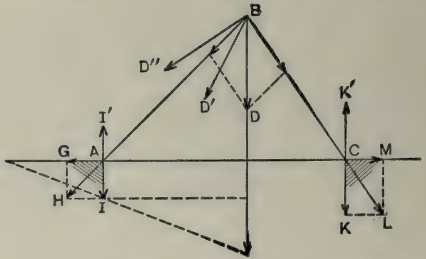

FIG. 3 (repeated).

Ex. 13. In Fig. 3, with BD as the applied force, find the reaction at C by moments about A.

Ex. 14. In Fig. 3, with BD as the applied force, find the reaction at A by moments.

Ex. 15. In Fig. 3, with BD' as the applied force and A fixed, find the reaction at C by moments about A.

Ex. 16. In Fig. 3, with BD' as the applied force and C fixed, find the reaction at A by moments.

Ex. 17. In Fig. 3, with BD'' as the applied force and C fixed, find the reaction at A by moments.

Ex. 18. In Fig. 4 check Av by moments about B.

Ex. 19. In Fig. 4 check Bb' by moments about D.

Ex. 20. Knowing one reaction in Fig. 4, check the other reaction by moments about any point in the plane of the diagram.

Fig. 4(a) represents an adjustable self-registering model of a crane, for use in connection with the solution of such problems as are solved in Fig. 4. The graphical solution may be checked by comparison with the stresses registered on the parts of the crane. The difference of readings on any part before and after loading should agree with the corresponding stress as determined by the graphical solution.

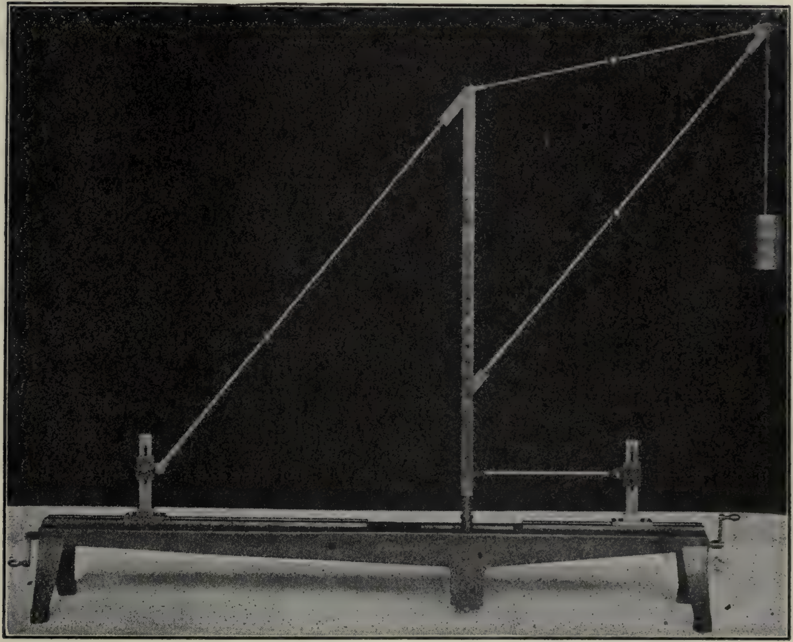

FIG. 4a

Ex. 21. Fig. 9. A uniform beam AB leans against a wall B. Friction on the floor prevents the foot of the beam from sliding, but no friction is developed on the wall. Construct the equilibrium.

The problem consists of finding the reactions at A and B.

The pressure on the wall at B must be normal to the wall, since a frictional surface can produce no resistance parallel to itself. The reaction at B is therefore in the direction BD. The weight of the beam, W, acts at the middle of the beam, E. The reaction at A must pass through D, which is the interesction of the action lines of W and the reaction at B. The resultant of the reactions, De, must be equal and directly opposed to the weight, and this makes the reaction at A determinate and equal to $Ah=Dk$. The horizontal component of Ah is the friction Af, and the vertical component is $Ag=W$.

For the horizonal equilibrium, $Af+Dk'=0$.

For the vertical equilibrium, $Ag+EW=0$.

Ex. 21 *a*. In Fig. 9 find the reaction at B by moments about A.

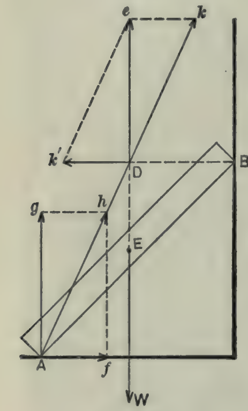

FIG. 9.

Ex. 22. Find the moment of the weight of a foot cube of silver with respect to a diagonal of one of the four vertical faces. Sp. g. 10.46.

Ex. 23. A body is weighed from each arm of a false balance, its apparent weights from the two arms separately being 81 and 100 grains. Find its true weight.

Ex. 24. Two uniform rods are attached to each other at right angles. One is inclined 30° from the vertical and mounted as an axis about which the other may turn. Find the moment of the weight of this rod when it is horizontal, in terms of its weight and length.

Ex. 25. A homogeneous rectangular granite block, 2×4×6 feet, has its 2×6 faces horizontal, and it weighs 8000 lb. Find the moments of its weight with respect to the diagonals of its vertical faces.

Ex. 26. A bar 6 ft. long and pivoted at the middle has a weight of 24 lb. hung at one extremity. What is the moment of the weight (*a*) when the bar is horizontal, (*b*) when it makes an angle of 30° below, and (*c*) of 60° above with the horizontal position?

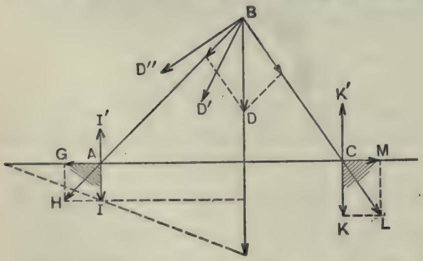

FIG. 3 (repeated).

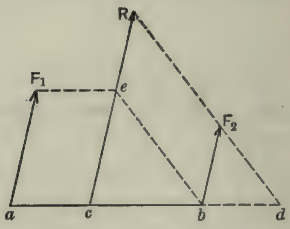

FIG. 10.

Ex. 27. If it is wished to upset a tall column by a rope of given length pulled from the ground, where should it be applied, if the length of the rope is, (1) equal to, (2) twice the height of the column?

Ex. 28. In Fig. 3, with *BD* given, construct *AI* and *CK* considered as parallel components of *BD*. (Transfer the point of application of *BD* to the horizontal tie *AC*, and use the method of parallel forces, Fig. 9.)

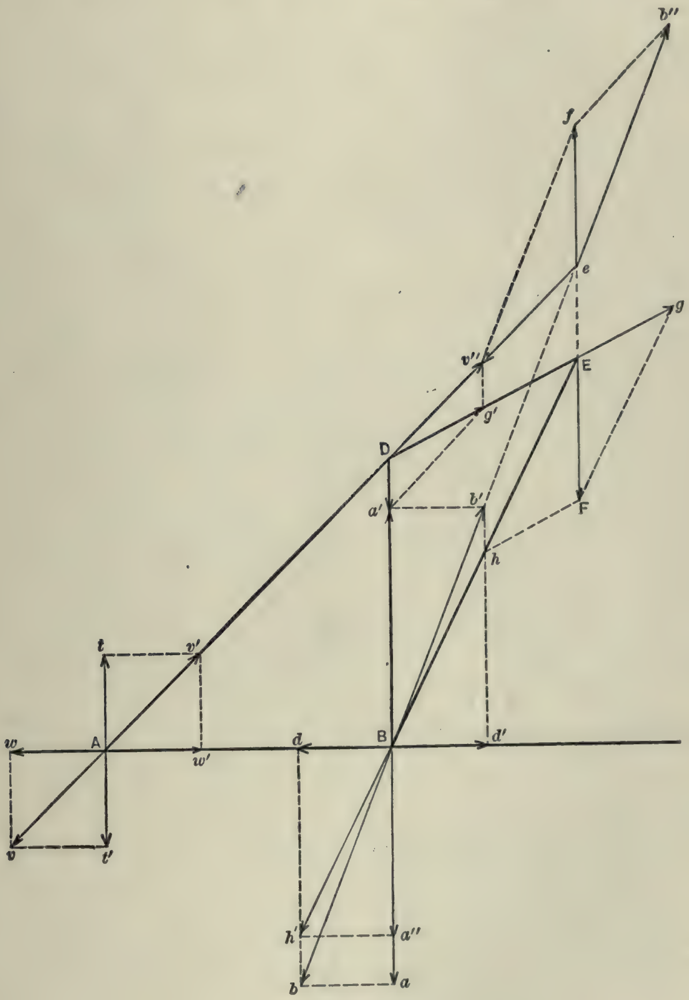

FIG. 11.

Ex. 29. In Fig. 11 construct the resultant of the vertic alreactions *At′* and *Ba′*.

Also find their resultant by Eqs. for coordinates of centroid (page 125, Hoskin's Mechanics).

Ex. 30. Check by the position of their resultant with respect to *EF*.

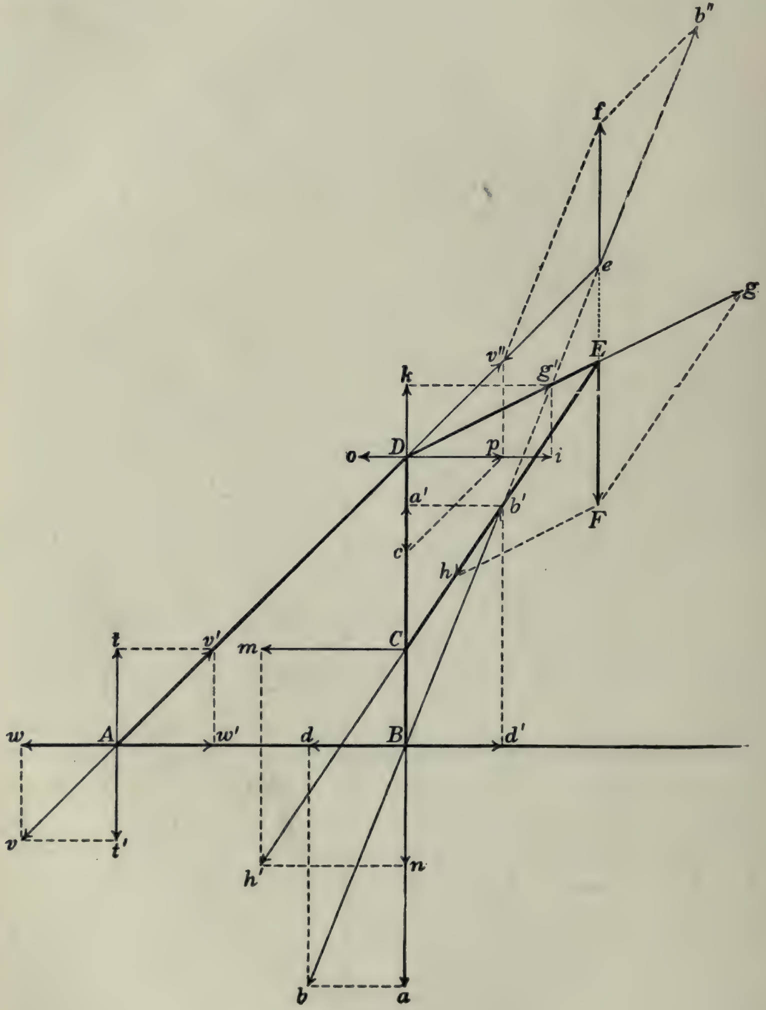

Fig. 12.

Check by moments.

Ex. 31. In Fig. 4 construct *At* and *Ba* as parallel components of *EF*.

Check by moments.

Ex. 32. In Fig. 9 find Af and Dk' by knowing that they constitute a couple equal and opposed to the couple (EW, Ag).

Ex. 33. Fig. 12 represents a derrick with its boom attached to the mast at a point some distance above the foot B. A bending force, Cm, is thus introduced, whose parallel components at B and D must be found in order to complete the solution. Make the necessary decomposition and follow the analysis of the diagram throughout. Note the difference between this problem and that of Ex. 8, Fig. 4.

Following is the analysis of the stresses transmitted from E:

The horizontal component of Eh, transmitted to C, is Cm. Cm has a bending effect on the upright, and has two parallel components, Do and Bd.

$Dg' = Eg$ has vertical and horizontal components Dk and Di. The algebraic sum of Do and Di is Dp, and the components of Dp on DB and AD are Dc and Dv''.

The horizontal and vertical components of the pull at A are Aw' and At. The reaction is Av, with components Aw and At'.

The horizontal and vertical components of the thrust at B are Bd and $Ba = Cn + Dc + Dk$, the thrust is Bd, and the reaction is Bb' with components Bd' and Ba'.

The resultant of the reactions at A and B, transferred to their common point e, is ef, equal and directly opposed to the load EF.

The horizontal pull at A balances the horizontal push at B. The resultant of the upward pull at A and the downward push at B is the load EF. Show this.

To check the stress on the guy by moments we have the following:

$$\left.\begin{array}{l}\text{Moment of the load about } B \text{ is } 12\times 16 = 192.\\ \text{Moment of } Dv'' \text{ about } B \text{ is } 8\sqrt{2}\times 24\sqrt{\tfrac{1}{2}} = 192.\end{array}\right\}$$

The stress at D may also be found by first constructing the resultant of Do and Dg', and then finding the components of this resultant in the directions DB and De. Or, the components of Do and Dg' may be constructed separately, and the algebraic sum of the components in each direction will give the stress in that direction.

Ex. 34. Fig. 13 (letter the figure A where the rope is attached to the bridge, B at the hinge, C at the point of the rope directly above B) shows the outline of a drawbridge AB, hinged at B and supported at A by a rope. W_1 is the weight of the bridge and W_2 a weight placed on the bridge.

Construct the tension on the rope and the reaction at B, and show that the horizontal components of the reactions at B and C constitute a couple in equilibrium with the couple formed by the vertical reaction at C and the vertical force at A.

Show also that the resultant of the reactions at B and C is equal and directly opposed to the resultant of W_1 and W_2.

Ex. 35. Fig. 14 indicates the principle of the common form of balance in which the weighing pans have parallel motions, controlled by a jointed parallelogram. Show that it is a matter of indifference where the weights are placed on the pans.

Assume a weight placed eccentrically on one of the pans, and find the couple which counteracts the moment of the weight with respect to one of the movable bearings of the pan.

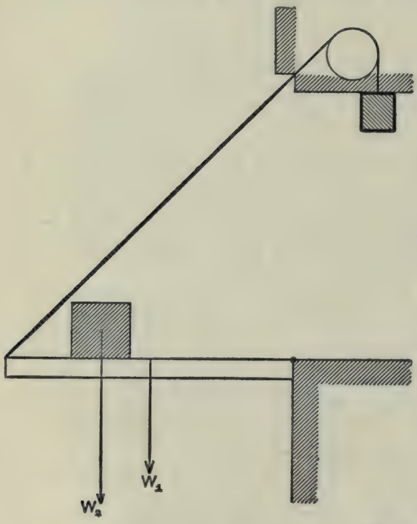

FIG. 13.

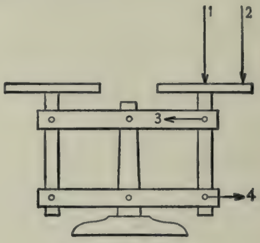

FIG. 14.

Show that the equivalent of this couple and the eccentric weight is an equal weight directly over the movable bearings.

Also show directly that the sum of the moments of the couple and the eccentric weight, with respect to one of the fixed bearings, is equal to the moment of an equal central weight.

Ex. 36. A particle on the concave surface of a sphere is repelled from the lowest point by a force which varies inversely as the square of the distance to the particle. Find the position of rest.

Assume the equilibrium as in Fig. 15, ρ being the radius of the sphere and c the chord at the end of which the particle is at rest.

The intensity of the repulsion at the distance unity is denoted by μ, and the normal reaction of the surface by N.

X and Z are taken in the plane of the vertical circle through m and the center of the sphere.

The equations of equilibrium are

$$R_x = \left(\frac{\mu}{c^2}\right)\left(\frac{x}{c}\right) + N\left(\frac{x}{\rho}\right) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$R_z = \left(\frac{\mu}{c^2}\right)\left(\frac{z}{c}\right) + N\frac{(\rho - z)}{\rho - W} = 0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (Eq. 1) we have $N = \frac{\mu\rho}{c^3}$, and this value in (2) gives

$$c = \left(\frac{\mu\rho}{W}\right)^{1/3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

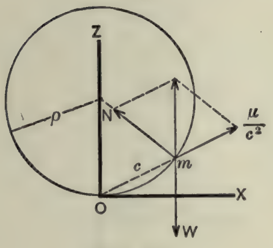

FIG. 15.

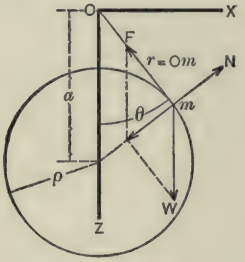

FIG. 16.

For any other intensity under the same law we have $c' = \left(\frac{\mu'\rho}{W}\right)^{1/3}$.

and hence

$$\mu : \mu' :: c^2 : c'^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

That is, the intensity varies directly as the cube of the distance at which the particle comes to rest. This is the principle of the pith-ball electroscope, which furnishes a rough means of measuring electric charges. The pith ball is attached to a center by a fine thread and repelled by an electric charge at the lowest point of the sphere whose radius is the distance from the fixed center to the center of the ball. Knowing the vale of μ and c, the value of μ' is found from (4) and the experimental value of c'.

Ex. 37. Let a particle m, of weight W, be held in equilibrium on a curve under the action of a given force directed toward the origin, as in Fig. 16. N is the normal reaction and r is the distance from the origin to m. Find the position of equilibrium.

The equilibrium is expressed by

$$R_x=\frac{Ndz}{ds}-F\sin\theta=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$R_z=-\frac{Ndx}{ds}-F\cos\theta+W=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eliminate N by multiplying the first of these equations by dx and the second by dz and then adding. We thus have

$$Wdz-F\sin\theta dx-F\cos\theta dz=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which (by $\sin\theta=\frac{x}{r}$, $\cos\theta=\frac{z}{r}$, $xdx+zdz=rdr$) reduces to

$$Wdz-Fdr=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The condition of the equilibrium may be found from this equation by using the equation of the curve to obtain the relation between dz and dr.

For example, the equation of the circle in Fig. 16 is $x^2+z^2-2az+a^2-\rho^2=0$, from which $adz=rdr$. This reduces (4) to

$$r=\frac{aF}{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

for the distance at which m is in equilibrium under the action of F and W.

Ex. 38. Are there any conditions of equilibrium in Fig. 16 other than that given by $r=\frac{aF}{W}$? If so, find them and give the conditions of the equilibrium.

Ex. 39. Find the curve on which a particle may be in equilibrium at any point under the action of its weight and a force of constant intensity directed toward the origin.

By integrating Eq. (4) Ex. 37, $Fdr-Wdz=0$, we have

$$Fr-Wz=\text{a constant}=C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Therefore the condition of equilibrium expressed by (4), Ex. 37, exists for all points of the curve whose equation is (1).

Making $z=r\cos\theta$ in (1) and solving for r we have

$$r=\frac{\frac{C}{F}}{1-\left(\frac{W}{F}\right)\cos\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Whatever may be the value of C we may write $C=Wp$, and thus we have for (2),

$$r=\frac{\left(\frac{W}{F}\right)p}{1-\left(\frac{W}{F}\right)\cos\theta}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

the equation of a conic section whose eccentricity is $\frac{W}{F}$.

The three cases are shown in Fig. 17. For the ellipse $F>W$, for the hyperbola $F<W$, and for the parabola $F=W$.

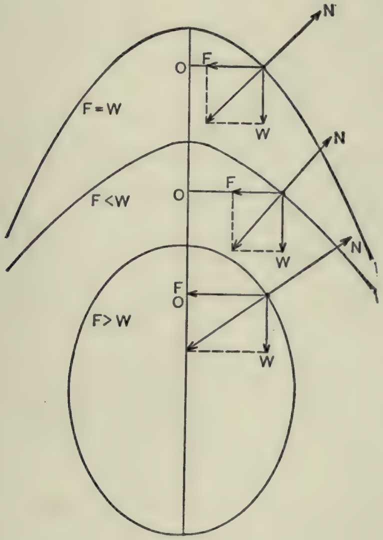

Fig. 17.

This problem suggests a method of finding the eccentricity of a conic section, as follows: From any point of the curve lay off any length on the normal, and make this length a diagonal of a parallelogram whose adjacent sides are drawn from the point, one of the focus and the other parallel to the principal axis. The ratio to these adjacent sides is the eccentricity of the curve.

Ex. 40. Give the possible position of equilibrium of the following homogeneous bodies of geometrical form, resting on a smooth horizontal plane: prolate spheroid; oblate spheroid; ellipsoid of three unequal axes; sphere.

Ex. 41. Find the center of gravity of a triangle.

A triangle may be conceived to be made up of lines parallel to a side, the center of gravity of each line being at its middle point. The locus of these middle points is a median of the triangle, and thus the center of gravity of the surface must be on each of the medians. It is therefore at their intersection.

Ex. 42. Find the center of gravity of the perimeter of a given polygon, of any given broken line, or of any group of right lines.

Ex. 43. Find the center of gravity of the area of any given polygon (group of triangles).

Ex. 44. Two given spheres are tangent externally. Find the center of gravity of their surfaces. Also of their volumes.

Ex. 45. An eccentric spherical shell is given by the exterior and interior diameters and the distance between the centers. Find the center of gravity of the shell.

Ex. 46. Show that the center of gravity of three equal weights one placed at the middle point of each side of a triangle, coincides with the center of gravity of the triangle.

Ex. 47. Find the center of gravity of a circular arc.

Take the radius of symmetry as the axis of x and the arc in the plane, XY, as in Fig. 18. Then $dl : dy :: r : x$, or $dW = dl = \frac{rdy}{x}$, and hence

$$= \frac{\left(\int_{x'}^{r} xdl\right)}{l} = \frac{\left(\int_{-y'}^{+y'} rdy\right)}{l} = \frac{r(2y')}{l}. \quad . \quad . \quad . \quad . \qquad (1)$$

That is, the center of gravity is on the radius of symmetry at a distance from the center equal to a fourth proportional to the arc, radius, and chord.

Ex. 48. Solve Ex. 47 by $\bar{x} = \frac{\left[\int_{x'}^{r} f(x)dx\right]}{l}$.

Ex. 49. Solve Ex. 47 by using $dl = \sqrt{(dx^2 + dy^2)}$ and $x^2 + y^2 = r^2$.

Ex. Find the center of gravity of a symmetrical arc of the cycloid.

Take the cycloid as in Fig. 19, and use the equation $l^2 = 8rx$. *Answer*, $\bar{x} = \frac{1}{3}x'$.

Ex. 51. Find the center of gravity of a triangle by integration.

Take a vertex at the origin and the opposite side parallel to Y, as in Fig. 20, and let $y'=ax$ and $y''=bx$ be the equations of the two other sides. Then we may write $dS=(y''-y')dx$, and

$$\bar{x}=\frac{\left[\int_0^{x'}(b-a)x^2dx\right]}{\left[\int_0^{x'}(b-a)xdx\right.}=\tfrac{2}{3}x',$$

$$\bar{y}=\frac{\left[\int_0^{x'}\tfrac{1}{2}(b^2-a^2)x^2dx\right]}{\left[\int_0^{x'}(b-a)xdx\right]}=\tfrac{2}{3}x'\frac{(a+b)}{2}.$$

This result agrees with that obtained in Ex. 41, the medians intersecting at two-thirds of the distance from any vertex to the middle of the opposite side.

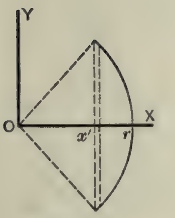

FIG. 18.

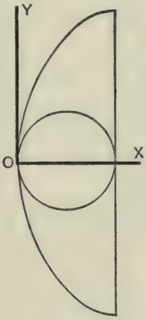

FIG. 19.

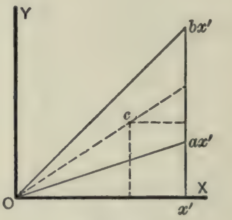

FIG. 20.

Ex. 52. Find the center of gravity of a circular sector.

The sector may be conceived to be made up of infinitesimal triangles having a common vertex at the center, with radii as sides. The center of gravity of each of these indefinitely small triangles is on its mid-radius at a distance from the center equal to two-thirds of its length (Ex. 51), and the locus of these centers is an arc having two-thirds of the length and radius of the arc of the given sector. The weight of the given sector may thus be supposed concentrated uniformly on this locus, whose center of gravity therefore coincides with the center of gravity of the sector. Therefore (Ex. 47) the center of gravity of a circular sector is on the mid-radius at a distance from the center equal to two-thirds of a fourth proportional to the arc, radius, and chord.

Ex. 53. Find the center of gravity of the lateral surface of a right cone or pyramid.

The cone or pyramid may be supposed to be made up of lateral sections parallel to the base, having weights proportional to their distances from the vertex. The locus of the centers of gravity of these sections is the right line from the vertex to the center of gravity of the base section. Take this line as the axis of x and the origin at the vertex.

The center of gravity of the cone or pyramid is the same as for the right line joining the vertex with the center of gravity of the base, assuming that the density of this line varies directly as the distance from the vertex.

This result may also be inferred directly from Ex. 51, since the elements parallel to the base vary according to the same law for the cone, pyramid, and triangle: also from the division of the surface into infinitesimal triangles with a common vertex, as in Ex. 52.

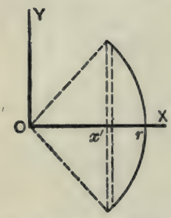

FIG. 18 (repeated).

Ex. 54. Find the center of gravity of a circular segment. Fig. 18.

Let the origin be at the center, and take the mid-radius as the axis of x. Then $dS=2ydx=2(r^2-x^2)^{1/2}dx$, and

$$\bar{x}=\frac{\left[\int_{x'}^{r}2(r^2-x^2)^{1/2}xdx\right]}{S}=\frac{[\frac{2}{3}(r^2-x'^2)^{3/2}]}{S}=\frac{\frac{2}{3}y'^3}{S}.$$

For S we have, either by $2\int_{x'}^{r}ydx$, or by geometrical analysis of the figure, $r^2\cos^{-1}\left(\frac{x'}{r}\right)-x'y'$. The area of the sector is $r^2\cos^{-1}\left(\frac{x'}{r}\right)$ and that of the triangle is $x'y'$.

Ex. 55. Assume $y^2=2px$.

a. Find the center of gravity of the area bounded by the axis of x, the parabola, and one ordinate. (Half the area of a parabolic arch, divided symmetrically.)

Answer, $\bar{x}=\frac{3}{5}x'$; $\bar{y}=\frac{3}{8}y'$.

(*b*) Find the center of gravity of the area bounded by the axis of y, the parabola, and one abscissa. (The area of a parabolic spandrel.)

Answer, $\bar{x}=\frac{3}{10}x'$; $\bar{y}=\frac{3}{4}y'$.

(*c*) Check (*a*) and (*b*) by the relations to the center of gravity of the rectangle $x'y'$.

Ex. 56. Find the center of gravity of a symmetrical half of an elliptical area.

Answer, $\bar{x}=\frac{\frac{4}{3}a}{\pi}$, or $\bar{y}=\frac{\frac{4}{3}b}{\pi}$.

Check the corresponding value for a semicircle by the rule given in Ex. 52.

Ex. 57. Find the center of gravity of the surface of a spherical zone.

Take the radius of symmetry as the axis of x, with the origin at the center of the sphere. Then dS may be obtained from the circle $x^2+y^2=r^2$, whose revolution about X generates the sphere. The differential of the surface may be taken as the surface generated by the differential of the revolving arc. Thus, Fig. 21,

$$dl : dx :: r : y,$$

$$ydl = rdx,$$

$$dS = 2\pi y dl = 2\pi r dx.$$

$$\bar{x} = \frac{\left[\int_{x'}^{x''} 2\pi r x dx\right]}{\left[\int_{x'}^{x''} 2\pi r dx\right]} = \frac{(x'+x'')}{2}.$$

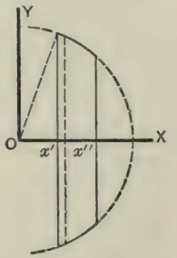

FIG. 21.

That is, the center of gravity is on the radius of symmetry and midway between the bases.

Ex. 58. Find the center of gravity of the volume of a paraboloid of revolution.

Take the axis of revolution as the axis of x, and $y^2=2px$ as the equation of the generating curve of the parabolic surface. Then $dV=\pi y^2 dx = 2p\pi x dx$, and

$$\bar{x} = \frac{\left[\int_0^{x'} 2p\pi x^2 dx\right]}{\left[\int_0^{x'} 2p\pi x dx\right]} = \tfrac{2}{3}x'.$$

Ex. 59. Find the center of gravity of the portion of a spheroid (solid) bounded by planes normal to the axis.

Take $y^2 = \frac{(2ax-x^2)b^2}{a^2}$ as the equation of the generating ellipse.

Ex. 60. Find the center of gravity of a solid right cone or pyramid.

Following the method of Ex. 53 this problem may be solved by a single integration by taking the origin at the vertex and the axis of x passing through the center of gravity of the base. Then the weight of a section parallel to the base varies as x^2, and $dW = cx^2dx$; hence

$$\bar{x} = \frac{\left[\int_0^{x'} cx^3dx\right]}{\left[\int_0^{x'} cx^2dx\right]} = \tfrac{3}{4}x'.$$

Ex. 61. Find the surface and volume of a sphere by the theorems of Pappus.

$$S = 2\pi\bar{x}l = (2\pi)\left(\frac{2r}{\pi}\right)(\pi r) = 4\pi r^2,$$

$$V = 2\pi\bar{x}S = (2\pi)\left(\frac{4r}{3\pi}\right)(\tfrac{1}{2}\pi r^2) = \tfrac{4}{3}\pi r^3.$$

Ex. 62. Find the area and volume of a right cone with a circular base.

Ex. 63. Find the area and volume of a right cylinder with a circular base.

Ex. 64. Find the volume of a paraboloid of revolution, the equation of the generating parabola being $y^2 = 2px$, as follows:

(*a*) Revolution about X.
(*b*) Revolution about Y.
(*c*) Revolution of the spandrel about X.
(*d*) Revolution of the sprandrel about Y.
(*e*) Revolution of the sprandrel about its limiting abscissa.

Ex. 65. Find the area of the surface generated by revolving a cycloid about its base.

Ex. 66. Find the area of the surface generated by revolving a cycloid about the tangent parallel to the base.

Ex. 67. Find the volume generated by the area of cycloid revolving about its base. $\left(\bar{x} = \frac{5r}{6}, \ S = 3\pi r^2.\right)$

Ex. 68. Take (2,63) as a center of moments, with component axes parallel to X and Z. Take a force of 2 units acting parallel to Y, negatively, at (12,68). Represent the component moments and the resultant moment. (Cross-section proper.)

Ex. 69. Assume (22,56) as a center of moments. A force of 4 at (19,56) acts perpendicular to and toward the paper, and a force of 8 acts at (22,54) perpendicular to and from the paper. Construct the component and resultant moments, and check by finding the moment of the resultant.

Ex. 70. Center of moments at (22,50).

Force 3 at (26,50), perpendicular to and toward the paper.

Force 4 at (22,46), perpendicular to and from the paper.

Construct and check the parallelogram of moments.

Ex. 71. Assume a center of moments at (8,38). A force of 2 at (4,41) and a force of 2 at (8,33), both perpendicular to and from the paper. Apply the parallelogram of moments, and check the resultant moment by the moment of the resultant.

Ex. 72. A tie-rod (2,28) (12,28) and a strut (2,18) (12,28) support a weight represented by (12,28) (12,22). Construct the pull on the tie and the push on the strut.

Ex. 73. Two inclined struts, (2,9) (7,17) and (7,17) (12,9), support a weight (7,17) (7,4). Construct the thrust on each strut.

Ex. 74. A cord (25,36) (29,28) (37,36) supports a weight (29,28) (29,18). Construct the pull on each branch of the cord.

Ex. 75. A cord (15,10) (25,10) (33,16) supports a weight (25,10) (25,4). Construct the pull on each branch of the cord.

Ex. 76. Apply Eqs. of centroid (Hoskins, p. 125) to find resultant and center of the following systems of forces; (3,63) (6,67), (5,62) (2,58), (13,68) (7,60), (4,53) (13,65).

Ex. 77. Find graphically the resultant of the forces (20,60) (20,66) and (30,60) (30,64), and check by proportion.

Ex. 78. Two homogeneous solid spheres are fastened together in contact.

The radii are 4 and 8, and densities are 4 and 1, respectively.

They are suspended by a cord fastened at the point of contact.

Construct the position of equilibrium.

Ex. 79. The resultant of two forces is (28,39) (28,27), and one component is (34,39) (34,42). What is the other component?

Ex. 80. Assume ground line (0,35) (20,35); upright (13,35) (13,50); tie (13,50) (19,50); strut (19,50) (13,41); guy (3,35) (13,50); force (19,50) (19,42$\frac{1}{2}$). Make complete diagram of stresses.

Ex. 81. Assume ground line (20,35) (40,35); upright (30,35) (30,50); tie (30,48) (36,48); strut (36,48) (30,36); guy (20,35) (30,50); force (36,48) (36,44). Make complete diagram of stresses.

Ex. 82. Assume ground line (0,30) (20,30); upright (14,30) (14,45); tie (14,45) (22,49); strut (22,49) (14,33); guy (4,30) (14,40); force (22,49) (22,43). Make a complete diagram of stresses.

Ex. 83. Assume ground line (20,30) (40,30); upright (30,30) (30,45); tie (30,45) (38,43); strut (38,43) (30,31); guy (20,30) (30,45); force (38,43) (38,39). Make complete diagram of stresses.

Ex. 84. Find the C. G. of the following group lines:

Circumference, center (13,15) and radius 3.
Circumference, center (33,15) and radius 5.
Right line, (16,15) (28,15).

Ex. 85. Apply the principle of the center of mass to finding the center of mass of two homogeneous solid spheres in contact, one sphere of radius 1 and density 16, the other of radius 2 and desnity 1.

Ex. 86. Find the C. G. of an arc of 60°.

Ex. 87. Find the C. G. of the arc of a semicircle.

Ex. 88. Find the C. G. of a circular sector of 60°.

Ex. 89. Find the C. G. of a right line whose density varies directly as the distance from one end.

Ex. 90. Find the C. G. of a right line whose density varies directly as the square of the distance from one end.

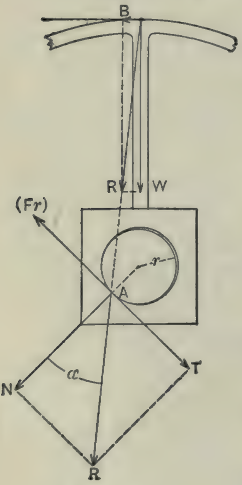

FIG. 22.

Ex. 91. Show that the volume of a spheroid is either $\frac{4}{3}\pi ab^2$, or $\frac{4}{3}\pi a^2b$, a and b being semi-axes of the axial section.

Ex. 92. Draw the circles, $c=(20,12)$, $r=6$; $c=(17,12)$, $r=3$; and find the C. G. of the area between the two circumferences.

Ex. 93. Find the friction on a journal turning under the action of a known resultant.

Let Fig. 22 represent the section of a journal and its bearing, in contact at A, supporting the weight W and turning under the belt pull B. The clearance between the journal and its box is exaggerated in the figure.

The resultant of B and W is R, and its point of application in the journal-box is at A. Equilibrium requires T, the tangential component of R, to be equal to the friction, as indicated in the figure. Then α is the angle of friction, assuming the surface of contact to be in the tangent plane at A. Then

$$R=\sqrt{N^2+T^2}=\sqrt{N^2+(Fr)^2}=\sqrt{N^2+f^2N^2}=N\sqrt{1+f^2}, \quad . \quad (1)$$

$$(Fr)=fN=\frac{fR}{\sqrt{1+f^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

from which the friction may be obtained directly from R without finding N. The moment of the friction of then obtained by multiplying the friction by the radius of the journal.

The same result is obtained from the equilibrium of moments, as follows:

$$(Fr)r = R(r \sin \alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$(Fr) = R \sin \alpha = \frac{R \tan \alpha}{\sqrt{1+\tan^2 \alpha}} = \frac{fR}{\sqrt{1+f^2}}, \quad . \quad . \quad . \quad (4)$$

Ex. 94. Find the moment of the friction on a circular pivot.

A circular pivot is shown in vertical and horizontal sections in Fig. 23. Let S be the bearing surface of the pivot, and p the normal pressure per unit area.

We may take for dS an elementary ring of radius r and width dr, and then $dS = 2\pi r dr$. The friction on this element is $fp2\pi r dr$, and its moment is $fp2\pi r^2 dr$. Then we have for the entire moment of friction,

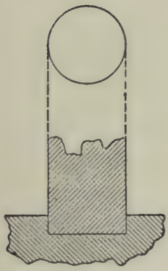

FIG. 23.

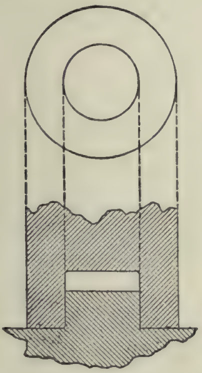

FIG. 24.

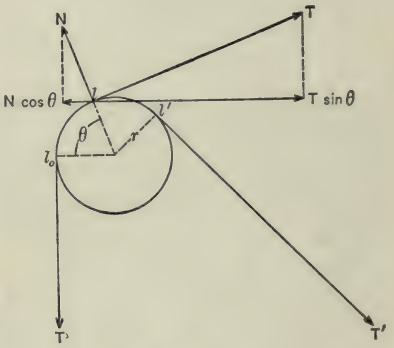

FIG. 25.

$$M_f = \int_0^a fp2\pi r^2 dr = \tfrac{2}{3} fp\pi a^3. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The entire friction is $(Fr) = fp\pi a^2$, and hence

$$M_f = (Fr)\tfrac{2}{3}a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That is, the mean lever of friction for a circular pivot is two-thirds of the radius.

Ex. 95. Find the moment of friction on a concentric ring pivot. Fig. 24.

Ex. 96. Analyze the moment of friction on an eccentric ring pivot.

Ex. 97. A belt over a pulley is assumed to be on the point of slipping, thus developing the entire friction between the pulley and the belt. Find the ratio of the limiting tensions, and the friction.

Assume the notation in Fig. 25. The arc enveloped by the belt is l_0ll', T_0 and T' are the limiting tensions, T the tension at any point, as at l, and N is the normal pressure at any point per unit of length of the belt, or arc of contact. T and N increase from l_0 to l', the belt bordering on motion over the pulley from l_0 around to l'.

Without friction the tensions would be equal throughout, and hence the difference between the tensions at any two points, as l_0 and l, is equal to the friction from l_0 to l. In other words the friction is a measure of the tension. The friction on dl is therefore the differential of the tension, and we have

$$dT=fNdl. \qquad \qquad (1)$$

This may be put in an integrable form by finding N in terms of T. To do this we may suppose T to be held in equilibrium by T_0 and the friction between l_0 and l. Assume no friction from l_0 to l; this is the same as saying $T_0=T$ and N is constant over the arc l_0l. Then we would have for the equilibrium of the components normal to T_0, from l_0 to l,

$$\int_{l_0}^{l} Ndl \cos\theta + T\sin\theta = 0, \qquad \qquad (2)$$

Integrating with N constant between 0 and θ and making $dl=rd\theta$ we have, numerically,

$$rN\sin\theta = T\sin\theta, \qquad \qquad (3)$$

from which

$$N=\frac{T}{r}. \qquad \qquad (4)$$

That is, the normal pressure per unit length at any point is equal to the tension divided by the radius.

Substituting $\frac{T}{r}$ for N in (1), we have

$$\frac{dT}{T}=\left(\frac{f}{r}\right)dl, \qquad \qquad (5)$$

and by integration,

$$\log_e T=\frac{fl}{r}+\log_e C. \qquad \qquad (6)$$

Making $l=0$ we have $C=T_0$, and

$$\frac{T}{T_0}=e^{\frac{fl}{r}}, \quad \ldots\ldots\ldots\ldots \quad (7)$$

$$\frac{T'}{T_0}=e^{\frac{fl'}{r}}, \quad \ldots\ldots\ldots\ldots \quad (8)$$

$$(Fr)=T'-T_0. \quad \ldots\ldots\ldots\ldots \quad (9)$$

Thus we have the means of solving the problem completely when f and one tension are known.

For the general case—which includes a rope wrapped around a cylinder, as in the working of a capstan—we may represent the number of complete turns of the rope by n and write $fn2\pi$ for the exponent in (8), the equation taking the form,

$$\frac{T'}{T_0}=e^{fn2\pi}. \quad \ldots\ldots\ldots\ldots \quad (10)$$

The rapid increase of T with n in equation (10) explains how the slack line from a capstan can be handled by one man while the strain on the working end of the line is very great; also the effectiveness of the snubbing-post in producing a great resistance by the application of a small force to the free end of the snubbing-line.

Ex. 98. The driving part of a belt is working under a pull of 400 lb., the slack and driving parts being parallel and $f=0.4$. What is the pull on the slack if it is running under the least possible strain?

Ex. 99. What coefficient of friction is necessary to snub a stress of 5000 lb. with two turns of the line around the post and a pull of 100 lb.? Why is this independent of the diameter of the post?

Ex. 100. If a body on an inclined plane be bordering on motion down the plane, show that the suspended weight may be increased by twice the friction without producing motion up the plane.

Ex. 101. Assume a uniform beam inclined between a horizontal floor and a vertical wall, as in Fig. 26, there being friction on both wall and floor. Find the inclination of the beam when it is on the point of sliding.

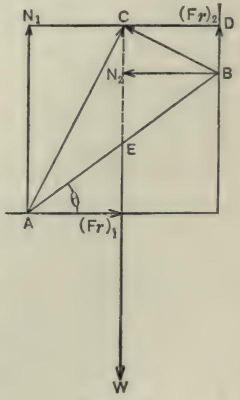

FIG. 26.

(*a*) **Analytical Solution.** For the horizontal equilibrium $(Fr)_1 + N_2 = 0$, and for the vertical equilibrium $W + N_1 + (Fr)_2 = 0$. With f known we have $(Fr)_1 = fN_1$ and $(Fr)_2 = fN_2$. Thus we have four equations to solve for the two frictions and the two normal reactions.

Then $\tan\theta$ may be found from the equilibrium of moments with respect to B,

$$N_1(l\cos\theta) = (Fr)_1(l\sin\theta) + W\tfrac{1}{2}l\cos\theta, \quad . \; . \; . \quad (1)$$

l being the length of the beam.

From (1) we have

$$\tan\theta = \frac{(N_1 - \frac{1}{2}W)}{(Fr)_1}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

The solution is thus seen to be independent of the length of the beam.

(*b*) **Graphical Solution.** The scale of the diagram and the length and weight of the beam are matters of indifference in finding θ. The proper proportions may be obtained as follows:

Assume any point on the floor, as A, for the foot of the beam, and lay off half the distance to the wall to represent $(Fr)_1$. Then lay off $N_1 = \frac{(Fr)_1}{f}$, and construct the resultant (AC) of $(Fr)_1$ and N_1. Extend N_1C to the wall at D and lay off $BD = f(CD) = f(Fr)_1 = fN_2$. Join A and B and lay off EW equal to $N_1A + DB$. This equilibrium diagram is correct to the scale for which EW represents the weight, but AB would generally be the length of the beam to a different scale. After θ has been constructed a diagram on any desired scale may be drawn.

The figure is constructed for $f = \frac{1}{2}$.

Ex. 102. Solve Ex. 101 when the center of gravity of the beam is given at some other point than the middle.

Ex. 103. A uniform ladder weighs 100 lb. and makes an angle of 30° with a vertical wall, the foot of the ladder resting on a horizontal floor. A man weighing 150 lb. climbs the ladder, and the ladder slips when the man is two-thirds of the way to the top. Find the coefficient of friction, assuming it to be the same for both wall and floor.

Ex. 104. Solve Ex. 101 with $f_1 = \frac{1}{2}$ and $f_2 = \frac{1}{3}$.

Ex. 105. Find the direction of the minimum force necessary to put a body in a condition bordering on motion up an inclined plane.

Let θ be the angle which the force makes with the plane, and i the inclination of the plane with the horizontal. Then for equilibrium bordering on motion up the plane we have

$$F\cos\theta = W\sin i - fF\sin\theta + fW\cos i, \quad . \quad . \quad . \quad . \quad (1)$$

and solving for F,

$$F = \frac{(W\sin i + fW\cos i)}{(\cos\theta + f\sin\theta)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The numerator of this value of F is a constant, and hence for F a minimum we must have $(\cos\theta + f\sin\theta)$ a maximum. Therefore

$$\frac{d(\cos\theta + f\sin\theta)}{d\theta} = f\cos\theta - \sin\theta = 0, \quad . \quad . \quad . \quad . \quad (3)$$

from which

$$\tan\theta = f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This value of $\tan\theta$ in the second differential coefficient gives

$$\frac{d(f\cos\theta - \sin\theta)}{d\theta} = -f\sin\theta - \cos\theta = -\sqrt{1+f^2}, \quad . \quad . \quad (5)$$

showing a maximum value of $(\cos\theta + f\sin\theta)$, and therefore a minimum value of F. That is, F is a minimum when its angle with the plane is the angle of friction.

Ex. 106. Within what limits of i is it possible to move a body up an inclined plane by applying the force horizontally?

The equilibrium is expressed by

$$F\cos i = W\sin i + fW\cos i + fF\sin i, \quad . \quad . \quad . \quad (1)$$

from which

$$F = \frac{(W\sin i + fW\cos i)}{(\cos i - f\sin i)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

($\sin i$ is + from 0 to 90°; so is $\cos i$).

F is real from $i=0$ to $i=90^\circ-\alpha$, the superior limit of i being reached when the denominator of (2) is zero. We then have

$$\cos i - f\sin i = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$f = \cot i = \tan\alpha, \quad . \quad . \quad . \quad . \quad (4)$$

$$i = 90^\circ - \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$i > (90^\circ - \alpha) \text{ gives } F \text{ negative.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

That is, a horizontal force cannot move a body up an inclined plane when the angle of the plane with the vertical is equal to or less than the angle of friction.

Ex. 107. The equilibrium of a wedge bordering on motion under a force normal to its back and the pressure on its faces. Fig. 27.

Taking the wedge to be isosceles and its angle ω, we have for the general equation of equilibrium, depending on the sign of the friction (the inward direction being positive),

$$F=2N\sin\tfrac{1}{2}\omega\pm 2fN\cos\tfrac{1}{2}\omega. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

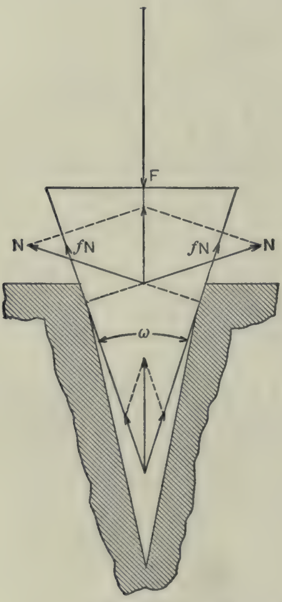

FIG. 27.

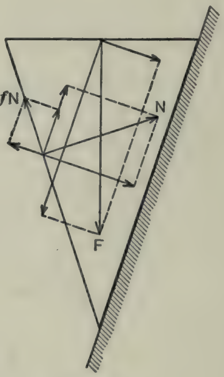

FIG. 28.

For $\omega=0$, $F=\pm 2fN$, requiring the same numerical value of F for motion inward and outward.

For $F=0$ we have $f=\tan\frac{1}{2}\omega=\tan\alpha$, or $\omega=2\alpha$, and the wedge is bordering on motion outward.

For $\omega<2\alpha$ we have $\tan\frac{1}{2}\omega<\tan\alpha$, or $\tan\frac{1}{2}\omega<f$, and hence $\sin\frac{1}{2}\omega<f\cos\frac{1}{2}\omega$; both values of F are real, one positive and one negative, and it requires a finite force either to push the wedge in or pull it out.

For $\omega>2\alpha$ we have $\tan\frac{1}{2}\omega>\tan\alpha$, or $\tan\frac{1}{2}\omega>f$, and hence $\sin\frac{1}{2}\omega>f\cos\frac{1}{2}\omega$. Therefore F is always positive for $\omega>2\alpha$; that is, it requires a force acting inward to hold the wedge in place when the angle of the wedge is greater than twice the angle of friction.

To find the limit of ω within which the wedge may be driven in, take the equation of equilibrium parallel to the face of the wedge when bordering on motion inward. We may consider the wedge to be resting on one face under the action of the forces applied at the back and the other face, F, N, and fN, as in Fig. 28. We thus have for equilibrium parallel to the face on which the wedge is assumed to be resting,

$$F\cos\tfrac{1}{2}\omega - fF\sin\tfrac{1}{2}\omega - f(N\cos\omega - fN\sin\omega)$$
$$-fN\cos\omega N\sin\omega = 0, \quad \ldots\ldots\ldots \quad (2)$$

from which

$$F = \frac{[(1-f^2)N\sin\omega + 2fN\cos\omega]}{(\cos\tfrac{1}{2}\omega - f\sin\tfrac{1}{2}\omega)}. \quad \ldots \quad (3)$$

F therefore becomes infinite when $\cos\frac{1}{2}\omega - f\sin\frac{1}{2}\omega = 0$, giving the condition

$$f = \tan\alpha = \cot\tfrac{1}{2}\omega, \quad \ldots\ldots\ldots \quad (4)$$

from which $\alpha = 90° - \frac{1}{2}\omega$; or

$$\omega = 180 - 2\alpha. \quad \ldots\ldots\ldots\ldots \quad (5)$$

That is, when the angle of the wedge is as great as 180° minus twice the angle of friction the wedge cannot be driven in.

Ex. 108. **Equilibrium of a Cord.** Let a cord be fixed at its ends and acted on by forces at given points not continuous, and neglect the stiffness and weight of the cord.

(*a*) The form of the cord and the directions of the forces are given.

Let a and b, Fig. 29, be the fixed ends, and let the forces be vertical. One tension or one force is sufficient to make the problem determinate. Let the tension at b be given. The vertical force 1 is then determined by the parallelogram whose diagonal is $c1$. Then the tension on cd determines the parallelogram at d, and so on. Similarly if one of the forces be given, the tensions on the adjacent branches of the cord may be determined, and the other tensions are then found as before.

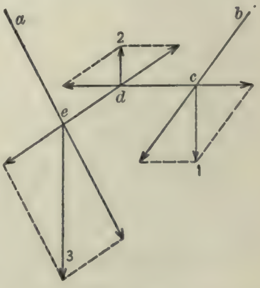

FIG. 29.

(*b*) The forces and the lengths of the branches are given.

In solving problems of this kind use the principles of equilibrium to state as many independent conditions as there are unknown tensions and directions. The solution is generally intricate, and it is often easier to solve graphically by trial.

In the above cases the principles of equilibrium apply to each point separately, and hence to the system as a whole. To make the general statement of equilibrium assume the origin at any point on the cord, and let the tension and direction angles of the cord at the cord at the origin be T_0 and θ_x, θ_y, θ_z. Let the tension and directions at any other point (x, y, z), be T and ϕ_x, ϕ_y, ϕ_z, and for any applied force between the origin and (x, y, z) take the usual symbols, F and α, β, γ.

Then we have

$$\left.\begin{aligned} T\cos\phi_x &= T_0\cos\theta_x + \Sigma F\cos\alpha, \\ T\cos\phi_y &= T_0\cos\theta_y + \Sigma F\cos\beta, \\ T\cos\phi_z &= T_0\cos\theta_z + \Sigma F\cos\gamma, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

(*c*) If forces act at all points of the cord so as to make the tensions vary by continuity, then the cord will assume the form of a curve, and Eq. (1) become

$$\left.\begin{aligned} \frac{Tdx}{ds} &= T_0\cos\theta_x + \Sigma F\cos\alpha, \\ \frac{Tdy}{ds} &= T_0\cos\theta_y + \Sigma F\cos\beta, \\ \frac{Tdz}{ds} &= T_0\cos\theta_z + \Sigma F\cos\gamma, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The line of which these are the differential equations is called a *funicular curve.*

Ex. 109. The fixed points of a cord 9 ft. long are 6 ft. apart in a horizontal plane. Two weights having the ratio of 1 to 2 are suspended so as to divide the cord into three equal branches. Find the form of equilibrium.

(*a*) *Graphical Solution by Trial.* - The resultant of the tensions on the extreme branches of the cord is equal and directly opposed to the resultant of the applied forces, and the point of application of the resultant weight is on the middle branch of the cord at the distance of 1 ft. from the point at which the greater force is applied. By trial make the extreme branches (produced) intersect on the vertical through the point of application of the resultant weight, as in

Fig. 30. The trial is made by drawing arcs with the fixed points as centers and the extreme branches as radii, and then adjusting the middle branch to the two arcs until the required condition is obtained.

(*b*) *Analytical Conditions.* The equilibrium is stated by the following equations:

$$\text{Horizontal, } t_1 \cos \theta_1 = t_2 \cos \theta_2 = t_3 \cos \theta_3, \quad \ldots\ldots \quad (1)$$

$$\text{Vertical, } t_1 \sin \theta_1 + t_3 \sin \theta_3 = W_1 + W_2, \quad \ldots\ldots \quad (2)$$

$$\text{Moments about } C,\ t_1\, 2 \sin (\theta_1 - \theta_2) = t_3 \sin (\theta_3 + \theta_2). \quad . \quad (3)$$

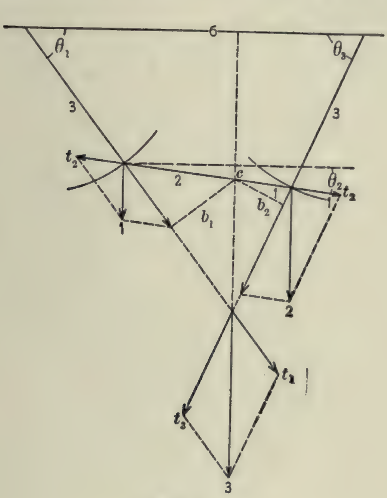

Fig. 30.

Two additional relations are obtained from the extreme horizontal and vertical dimensions of the figure formed by the cord, as follows:

$$\cos \theta_1 + \cos \theta_2 + \cos \theta_3 = 2, \quad \ldots\ldots \quad (4)$$

$$\sin \theta_1 + \sin \theta_2 = \sin \theta_3. \quad \ldots\ldots \quad (5)$$

We thus have six conditions for the three tensions and three angles.

Ex. 110. A flexible cord supports a uniform horizontal load. Find the equation of the form of the cord, assuming its weight to be negligible.

Assume the origin at the lowest point and the cord in the plane XZ. Then Eqs. (2), Ex. 108 become

$$\frac{Tdx}{ds} = T_0 \cos \theta_x = c, \qquad \ldots\ldots \quad (1)$$

$$\frac{Tdz}{ds} = \Sigma F. \qquad \ldots\ldots \quad (2)$$

That is, the horizontal component of the tension is constant, and the vertical component of the tension at any point is equal to the load between that point and the lowest point.

The load from O to x varies directly as x, and hence $\frac{Tdz}{ds} = ax$, a being the load per unit of x.

Eliminate T and ds from $\frac{Tdx}{ds} = c$ and $\frac{Tdz}{ds} = ax$, giving

$$axdx = cdz.$$

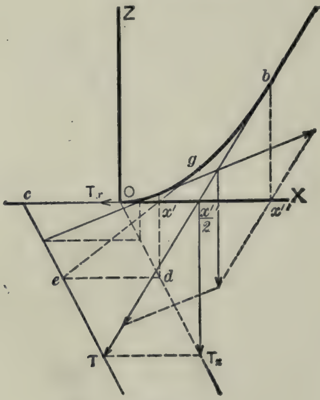

Fig. 31.

By integration we have

$$x^2 = \left(\frac{2c}{a}\right) z, \qquad \ldots\ldots \quad (3)$$

for the equation of the cord. Hence a flexible cord subjected to a uniform horizontal load takes the form of a parabola.

This fact is also shown by the construction of the tension at any point. In Fig. 31 take the origin at O, where the tension is horizontal. Take $a = 1$; then the load between O and x becomes equal to x with center of gravity at $\frac{1}{2}x$. The horizontal components of the tensions are equal, and, therefore, any part of the cord sustains the load immediately under that part. For example, the load between x' and x'' is held by the tensions at x' and x'', and the action lines of these two tensions must intersect on the vertical at $\frac{1}{2}(x'+x'')$, this vertical being the action line of the resultant of that part of the load between x' and x''. Therefore, taking one of the tensions at the origin, we see that the tension at x must intersect X at $\frac{1}{2}x$.

It is a property of the parabola that any tangent cuts the tangent through the vertex at a distance from the axis equal to one-half that of the point of tangency. Therefore the construction of the action lines of the tension is the construction of the tangents of a parabola.

The construction of the tension and the point of tangency for x'' in the figure is indicated by $T_x=c$, $T_z=x''$, and $T=$ the tension at the point of tangency, this point being at b, the intersection of the ordinate at x'' with the action line of T.

Knowing a and c the curve may be constructed by points as follows:

Represent the load from O to any value of x, laid off from the point $(\frac{1}{2}x, 0)$, and draw a right line through its extremity and the origin. Then draw a parallel line through the extremity of c, c being laid off to the left from the origin.

To construct any point of the curve, draw any vertical line cutting X and the inclined line through the origin, as at x' and d. Through d draw a horizontal cutting the parallel line to Od at e. Draw ex' prolonged to cut the vertical at $2x'$, thus giving g, which is a point of the curve.

Ex. 111. A cable subjected to a horizontal load of 2000 lb. per foot is suspended from towers 40 ft. above the lowest point of the cable, with a span of 120 ft. Find the tensions at the lowest and highest points, and the equation of form.

(*a*) To find c substitute the coordinates of the highest point in $x^2=\left(\frac{2c}{a}\right)z$, giving

$$(60)^2=\left(\frac{2c}{2000}\right)40,$$

from which

$$c=90{,}000 \text{ lb.}$$

(*b*) For the tensions at the highest points we have $T_z=120{,}000$ and $T_x=90{,}000$. Hence

$$T=\sqrt{[(120{,}000)^2+(90{,}000)^2]}=150{,}000 \text{ lb.}$$

(*c*) For the equation of form we have $\frac{2c}{a}=90$, and hence

$$x^2=90z.$$

Ex. 112. A cable subjected to a uniform horizontal load of 3000 lb. per foot is suspended from two points 25 and 36 ft. above the lowest point, with a span of 110 ft. Find the position of the lowest point, the tensions at the lowest and highest points, and the equation of form.

(Find the position of the lowest point by $x_1^2=\left(\frac{2c}{a}\right)25$, $x_2^2=\left(\frac{2c}{a}\right)36$, $x_1+x_2=110$.

Ex. 113. **The Catenary.** Fig. 32. The catenary is the curve taken by a uniform flexible cord under the action of its weight only. In this case the weight is proportional to the length of the cord, and hence we may take the unit length to have the unit weight. Then, with the coordinates measured from the lowest point, Eqs. (2), Ex. 108, we have

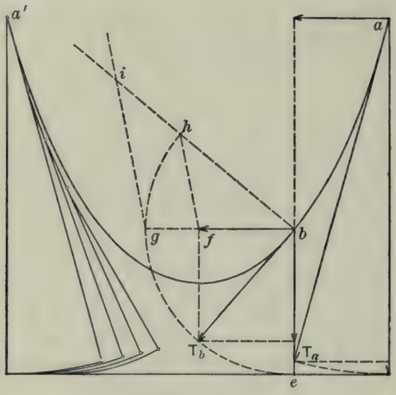

FIG. 32.

$$\frac{Tdx}{ds}=c, \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{Tdz}{ds}=\int_0^s ds=s. \quad \ldots\ldots\ldots \quad (2)$$

Eliminating T from these equations by

$$T=\sqrt{c^2+s^2}, \quad \ldots\ldots\ldots \quad (3)$$

we have

$$dx=\frac{cds}{\sqrt{c^2+s^2}}, \quad \ldots\ldots\ldots \quad (4)$$

$$dz=\frac{sds}{\sqrt{c^2+s^2}}. \quad \ldots\ldots\ldots \quad (5)$$

The equation of the catenary is obtained by integrating these two equations and eliminating s from the resulting values of x and z.

From (4) we have (Granville's Calculus, [21] p. 293),

$$x=c\log_e(s+\sqrt{c^2+s^2})+C=c\log_e\left[\frac{(s+\sqrt{c^2+s^2})}{c}\right]. \quad . \quad (6)$$

From (5) we have

$$z=\sqrt{c^2+s^2}+C=\sqrt{c^2+s^2}-c. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Solve (5) for s and substitute in (4), and we have for the equation of the catenary,

$$x=c\log_e\left[\frac{(\sqrt{z^2+2cz}+z+c)}{c}\right]. \quad . \quad . \quad . \quad . \quad (8)$$

The Directrix of the Catenary. From the values of T and z, (3) and (7), we have

$$T=z+c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

That is, the tension at any point is given by the ordinate$+c$ or by the ordinate when the axis of x is taken at the distance c below the lowest point. The axis of x is called the directrix of the catenary.

To make the directrix the axis of x, substitute $z-c$ for z in Eq. (8).

We thus have *

$$ce^{\frac{x}{c}}-z=\sqrt{z^2-c^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which reduces to

$$z=\tfrac{1}{2}c(e^{\frac{x}{c}}+e^{-\frac{x}{c}}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The corresponding form for the original origin, is

$$z=\tfrac{1}{2}c(e^{\frac{x}{c}}+e^{-\frac{x}{c}})-c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The tangent at any point may be constructed as follows:

Take the given point as a center and draw an arc tangent to the directrix; then draw the tangent to the curve through the point

---

* For Eq. (10) write

$$x=c\log_e\left\{\frac{[\sqrt{(z^2-c^2)}+z]}{c}\right\},$$

$$\frac{x}{c}=\log_e\left\{\frac{[\sqrt{(z^2-c^2)}+z]}{c}\right\},$$

$$e^{\frac{x}{c}}=\frac{\sqrt{(z^2-c^2)}+z}{c},$$

$$ce^{\frac{x}{c}}-z=\sqrt{(z^2-c^2)}. \quad \text{(Eq. (10) as above).}$$

where this arc cuts the ordinate which is at the distance c (toward the zero ordinate) from the ordinate of the given point.

Such a construction is shown at a and at b, together with the resolution of the tension into its horizontal and vertical components, which makes the construction obvious.

This suggests a method of approximating to the form of the curve when c and the span are given. Substitute c and the half span for x in (11) and thus find z for the highest point. This establishes the directrix and the highest and lowest points. Then starting at a' on the left branch, Fig. 32, draw the tangent as just explained. Then a point on the tangent slightly below a' may be taken as a point of the curve without great error. At this point draw another tangent and take another point as approximately on the curve. Continue the operation and take the envelope of the tangents so drawn as the approximate catenary.

The error accumulates toward the lowest point, but the method gives a reasonable approximation for merely illustrative use. Since the lowest point is known the error may be reduced by drawing a smooth curve near this graphical approximation and tangent to the known limiting tangents of the branch, thus making a second approximation to the true form. The error is exaggerated in the figure so as to show the method clearly.

The intrinsic equation of the catenary, and the construction of the radius of curvature. From the construction of the tension at any point we see that the intrinsic equation of the catenary is

$$s=c\tan\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

We therefore have for the radius of curvature,

$$\rho=\frac{ds}{d\phi}=\frac{c}{\cos^2\phi}=\frac{(c^2+s^2)}{c}. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The radius of curvature at the lowest point is therefore equal to c.

From (14) we have

$$\rho : \sqrt{c^2+s^2} :: \sqrt{c^2+s^2} : c, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and ρ may be constructed by this proposition, as follows:

Draw the direction of the radius, as bi, and lay off $bg=bh=(c^2+s^2)^{1/2}=T$. Then draw gi parallel to fh, giving the center of curvature at i, bi being the geometrical construction ρ.

Ex. 114. Find the ratio of the applied force to the resistance in the use of the *wheel and axle.*

The wheel and axle is represented in Fig. 33. This machine is essentially a continuous lever in which the force is applied by means of a rope around a wheel, while the load to be moved acts on a rope around the drum.

Let the machine be bordering on motion in the direction of the applied force, F, and let S be the resistance which the rope offers to bending around the drum. Then, representing the radii of the wheel, drum and journal by r_w, r_d, and r_j, respectively, the equilibrium of moments is expressed by

$$Fr_w - Wr_d - Sr_d - \left(\frac{Rf}{\sqrt{1+f^2}}\right)r_j = 0. \quad . \quad . \quad . \quad . \quad (1)$$

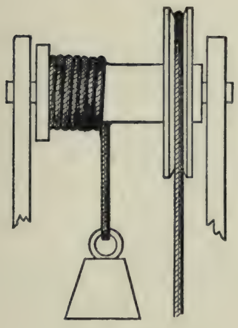

FIG. 33.

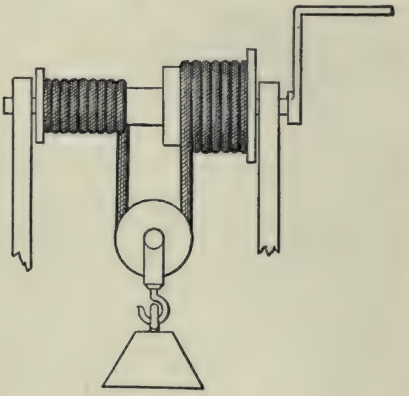

FIG. 34.

The stiffness of a rope has been found to vary with the load and with the radius of the wheel over which it is wound. Its value takes the form

$$S = \frac{(a+bW)}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which a is the resistance to bending over an arc of unit radius under no load, b is the stiffness for unit load and unit radius, W is the load, and r is the radius of the arc over which the rope is wound.

The resistance a is small and may be omitted in finding an approximate ratio of F to W. Then if we take $R = W + F$, omitting the effect of stiffness and friction in this estimate, we will have from (1),

$$\frac{F}{W} = \frac{\left(\frac{r_d + b + r_j f}{\sqrt{1+f^2}}\right)}{\frac{r_w - r_j f}{\sqrt{1+f^2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which approaches the limit

$$\frac{F}{W}=\frac{r_d}{r_w}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

as the stiffness and friction approach zero. At this limit F and W are inversely as their lever-arms, according to the simple relation of the lever.

Ex. 115. **The Differential Wheel and Axle,** or *Chinese windlass,* is shown in Fig. 34. In this form of the machine half the load, with a lever-arm equal to the radius of the smaller drum, acts with the applied force. The equilibrium, omitting the friction and the stiffness of the rope, is expressed by

$$Fr_w+W\tfrac{1}{2}r_1-W\tfrac{1}{2}r_2=0, \quad . \quad . \quad . \quad (1)$$

from which

$$\frac{F}{W}=\frac{\frac{1}{2}(r_2-r_1)}{r_w}. \quad . \quad . \quad . \quad . \quad (2)$$

This ratio diminishes as r_1 approaches r_2, but when $r_1=r_2$ the path of W becomes zero. By approaching this limit the *gain of power,* as measured by the ratio of W to F, may be made as great as desired.

In applying the power to either form of the wheel and axle a crank may be used instead of the rope and wheel.

Ex. 116. **The Differential Pulley,** Fig. 35, is a modified form of the differential wheel and axle. A continuous chain is wound around the three pulleys, the two upper pulleys, of different diameters, being fixed together on the same shaft. These two pulleys have recesses for the links, so that the chain cannot slip.

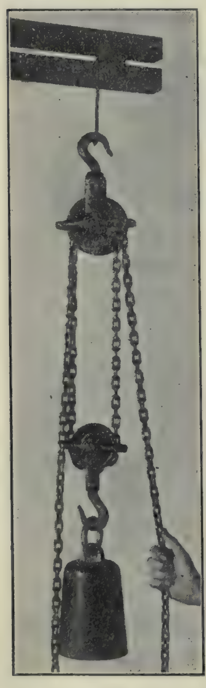

FIG. 35.

The power is applied to the larger pulley by pulling on the chain, and the stiffness of the chain is sufficient to prevent the weight from running down when the power is removed. The mechanical advantage is the same in principle as for the differential wheel and axle, and we have for the limiting ratio of F to W,

$$\frac{F}{W}=\frac{\frac{1}{2}(r_2-r_1)}{r_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Ex. 117. Find the ratio of the applied force to the resistance in the *block and tackle.*

The common form of the *block and tackle,* or *block and fall,* is shown in Fig. 38. The number of pulleys employed depends on the desired gain of power, and obviously the gain of power increases with the number of branches of the rope sustaining the load.

The pulleys in the upper block, which is attached to a fixed support, are called *fixed pulleys,* and those of the block attached to and moving with the load are called *movable pulleys.* The analysis of the problem includes the relation of the power to the load for each of these pulleys.

(*a*) *The Fixed Pulley.* The equation of equilibrium for a fixed pulley, Fig. 36, when the rope is bordering on motion in the direction of the applied force, is

$$Fr_p - Wr_p - Sr_p - (Fr)r_j = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

r_p being the radius of the pulley. This gives an evident loss of power for a fixed pulley, since F is greater than W and approaches W as the friction and stiffness approach zero. The purpose of the fixed pulley is simply to transmit the power in a direction other than that in which it is directly applied.

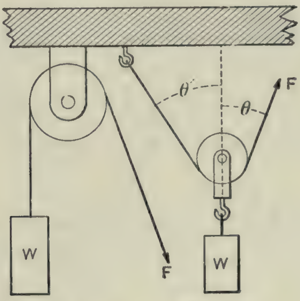

FIG. 36. FIG. 37.

(*b*) *The Movable Pulley.* Fig. 37 shows a movable pulley, with load and applied force. Assume the notation in the figure, T being the tension in the fixed branch of the rope.

For equilibrium bordering on motion in the direction of F we have

$$F \cos \theta + T \cos \theta' - W = 0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and eliminating T by

$$T = \frac{F - S - (Fr)r_j}{r_p}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

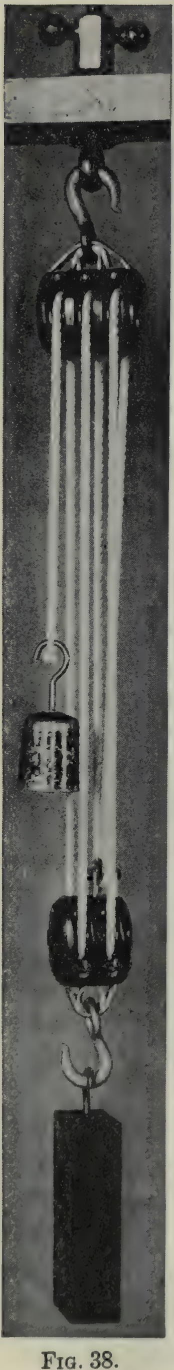

FIG. 38.

we have

$$F\cos\theta+\left[\frac{F-S-(Fr)r_j}{r_p}\right]\cos\theta'=W, \quad . \quad (4)$$

which approaches

$$\frac{F}{W}=\frac{\frac{1}{2}}{\cos\theta'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

as (Fr) and S approach zero, since $\theta=\theta'$ at this limit.

Let ϕ be the arc enveloped by the rope. Then we have for the chord of ϕ,

$$c=2r_p\sin\tfrac{1}{2}\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For zero values of S and (Fr), $\cos\theta=\sin\frac{1}{2}\phi$, and (106) reduces to

$$\frac{F}{W}=\frac{r_p}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

That is, the limit of the ratio of the power to the load is equal to the ratio of the radius of the pulley to the chord of the arc enveloped by the rope. There is therefore a gain of power when the arc enveloped by the rope is between 60° and 300°. The greatest gain is for $\phi=180°$, when $\frac{F}{W}=\frac{1}{2}$.

(*c*) *The Block and Tackle.* In the common block and fall we take the branches of the rope which support the load to be parallel, with the power applied to the *free end* or *fall.* Let the successive tensions be t_1, t_2, etc., beginning with the branch attached to one of the blocks. The tensions in two consecutive branches are related by the equation for the fixed pulley (1). The pulleys are assumed to be uniform in size and in the condition of the bearings, and we may take the friction and stiffness of the rope at any pulley to be proportional (approximately) to the tension of the branch which winds on the pulley. That is, we neglect a in Eq. (2), Ex. 114, and also the friction due to the difference of the tensions in the two branches separated by the pulley. For example,

$$t_2=t_1+\left\{bt_1+\left[(t_2+t_1)\left(\frac{f}{\sqrt{1+f^2}}\right)\right]\right\}r_j\div r_p,$$

and if in the second member of this relation we write t_1 for t_2 we shall have $t_2=t_1$ (1+a constant), say $t_2=ct_1$. Thus we have the series of relations,

$$\left.\begin{aligned} t_2&=ct_1,\\ t_3&=ct_2,\\ t_4&=ct_3,\\ t_n&=ct_{n-1},\\ F&=ct_n. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

n being the number of parallel branches supporting the load. From these we have

$$\left.\begin{aligned} t_3&=c^2t_1,\\ t_4&=c^3t_1,\\ t_n&=c^{n-1}t_1,\\ F&=c^nt_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In addition to this relation we have

$$W=t_1+t_2+\ .\ .\ .\ +t_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

or in terms of t_1 and c,

$$\begin{aligned} W&=t_1+ct_1+c^2t_1+\ .\ .\ .\ +c^{n-1}t_1\\ &=t_1(1+c+c^2+\ .\ .\ .\ +c^{n-1}). \quad . \quad . \quad . \quad . \quad . \quad (11) \end{aligned}$$

Eliminating t_1 from (9) and (11) the ratio of F to W reduces to

$$\frac{F}{W}=\frac{c^n}{1+c+c^2+\ .\ .\ .\ c^{n-1}}=\frac{(c-1)c^n}{(c^n-1)}. \quad . \quad . \quad . \quad (12)$$

The value of c approaches 1 as S and (Fr) approach zero, and the limiting ratio of F to W is therefore

$$\frac{F}{W}=\frac{1}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This is an obvious relation, since the n parallel branches of the rope would then support the load equally.

Ex. 118. Find the ratio of the power to the resistance in the common screw-jack.

Fig. 39 illustrates the common screw-jack.

The central cylinder is called the *newel*, the helical projection surrounding the newel is called the *thread*, and the vertical distance corresponding to one complete turn of the helix is called the *pitch*. In the standard square thread the depth and width of the thread are half the pitch, with sufficient clearance to work properly in the nut.

A force is applied to the lever, and the turning of the lever lifts a weight on the head of the jack, while the screw rises in the nut. For this purpose a rectangular thread is the best form for the screw. Other things being equal the normal pressure would be greater on the helicoidal surface of the standard triangular thread, since the lesser inclination of the bearing surface to the vertical would require a greater normal pressure to support the weight. The friction would, therefore, be less on the rectangular thread.

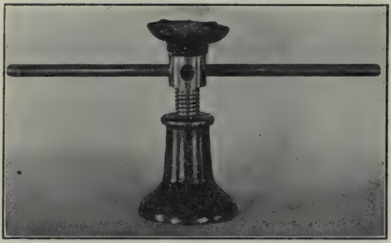

Fig. 39.

The angle of the helix with the axis of the screw diminishes as the helical element of the surface approaches the axis, and a rigid analysis of the forces acting would be very intricate; but, if we assume a small depth of thread relative to the diameter of the screw, we may take a mean helix to give the inclination of the helicoidal surface, and then suppose the forces to act as if the weight were to be moved up an inclined plane of this inclination.

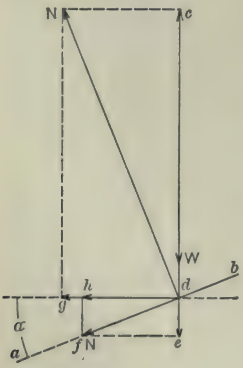

Fig. 40.

Assume the following notation:

α, the angle of the mean helix with the horizontal;
r, the radius of the mean helix;
W, the weight to be lifted;
N, the normal pressure, supposed to act at the mean helix;
F, the applied force, direction horizontal;
l, the lever-arm of F.

In Fig. 40 d is assumed as a point on the mean helix, and for the purpose of finding the ratio of F to W we may suppose the load, normal pressure, and friction to be concentrated at d. The inclination of the mean helix is indicated by ab, and the normal pressure is indicated so as to represent the pressure of the nut on the under surface of the thread while the screw is turning so as to rise in the nut. The load is represented by cW. Then for the vertical equilibrium bordering on motion of the screw upward in the nut we have

$$W=N\cos\alpha-fN\sin\alpha, \qquad (1)$$

and for the equilibrium of moments

$$Fl=N\sin\alpha r+fN\cos\alpha r. \qquad (2)$$

These give the following ratio of F to W:

$$\frac{F}{W}=\frac{(\sin\alpha+f\cos\alpha)r}{(\cos\alpha-f\sin\alpha)l}. \qquad (3)$$

As f approaches zero the limit of this ratio is

$$\frac{F}{W}=\frac{r\tan\alpha}{l}=\frac{2\pi r\tan\alpha}{2\pi l}. \qquad (4)$$

That is, the limit of the ratio of the applied force to the weight is the ratio of the pitch of the screw to the horizontal path of the applied force in making one turn.

Ex. 119. **The Differential Screw.** The gain of power in the common screw-jack increases as the pitch diminishes. The strength of the thread, however, decreases with the pitch, and beyond a certain limit the thread would be stripped from the newel under the desired load. This difficulty may be overcome by using a double screw of sufficient strength, with threads of different pitch, both right-handed or both left-handed and so arranged that the rise in one turn is the difference of pitch. Such a device is called a *differential screw.*

Design a differential screw-jack, and find the limiting ratio of the power to the load.

Ex. 120. A specimen of steel 6 in. long and 0.2 sq.in. cross-section is elongated 0.01 in. by a force of 10,000 lb. What is the modulus of elasticity?

At an elongation of 0.02 in. a permanent set begins. What is the elastic limit?

Ex. 121. A brass wire 3 ft. long and cross-section 0.01 sq.in. is elongated 0.036 in. by a weight of 150 lb. Find Young's Modulus.

Ex. 122. The Modulus of Elasticity of a steel rod is 30,000,000 lb.

Cross-section $= \frac{1}{2}$ sq. in. Length $= 10$ ft. What is the elongation under a pull of 50,000 lb.?

(Elastic limit $>$ 100,000 lb.)

Ex. 123. Young's modulus for a steel wire is 28,000,000 lb. per sq. in., and the elastic limit is 56,000 lb. per sq. in. The wire is 10 ft. long. How much can it be stretched without taking a permanent set?

Ex. 124. A steel wire 800 ft. long is suspended from one end. How much does it elongate under its own weight? ($E = 28{,}000{,}000$ lb. per sq. in. Wt. $=$ 504 lb. per cu. ft.)

Ex. 125. A specimen of steel of $\frac{1}{2}$ sq. in. cross-section is stretched one-twentieth of 1 per cent of its length by a force of 7000 lb. Find the modulus of elasticity.

Exs. 123 to 140, inclusive, will be found in "Notes on Graphical Statics."

Ex. 141. How many dynes in one poundal?

Ex. 142. How many poundals in 1 lb_w?

Ex. 143. How many grams in the C. G. S. gravitational unit of mass?

Ex. 144. Assume the velocities (or accelerations) (2,60) (5,70), (2,60) (10,60) and construct the resultant.

Ex. 145. Assume the velocities (or accelerations) (2,60) (5,70), (2,60) (10,60), (2,60) (12,55) and construct the resultant.

[Add (2,60) 12,55) to diagram of problem 144].

Ex. 146. One component velocity is (25,60) (30,68) and the resultant is (25,60) (40,68). Construct the other component.

Ex. 147. A's v is (30,50) (30,58), and B's v is (40,50) (32,42). How fast are A and B approaching each other?

What is their relative velocity?

Ex. 148. C's v is (5,50) (5,56), and D's v is (8,46) (18,46). How fast are C and D separating from each other?

What is their relative velocity?

Ex. 149. A is at (13,24) and B at (37,24), both moving toward (25,40) with $v = 10$.

Construct their rate of approach, and the apparent velocity of B as seen from A.

Ex. 150. A is at (15,12) with velocity (15,12) ($1\frac{1}{2}$,12), and B is at (21,4) with velocity (21,4) (21,14).

Find by construction

The velocity of approach.

The *apparent* velocity of A as seen from B.

The *apparent* velocity of B as seen from A.

Ex. 151. A train is running S. E. $35.3555\frac{m}{h}$. How fast is it going E.?

Ex. 152. *A* is northeast of *B*, *A* going north and *B* going east at the same rate.

(*a*) What is their rate of approach?

(*b*) In what direction does *A* appear to move as seen from *B*?

Ex. 153. *A* is northwest of *B*. *A* is going east 4 miles per hour and *B* is going south 2 miles per hour. How fast is *A* approaching *B*, and in what direction does *B* appear to move as seen from *A*?

Ex. 154. Assume that rain-drops are falling vertically $33\frac{f}{s}$. At what angle from the vertical do they appear to fall as seen by a man walking $3\frac{3}{4}\frac{m}{h}$?

Ex. 155. A N. 30° E. wind blows $10\frac{m}{h}$. What is the apparent direction of the wind to a man walking $5\frac{m}{h}$ E.?

Ex. 156. A S.W. wind blows $35.3555\frac{m}{h}$. How fast must a train travel eastward to make the wind appear to blow from the S. E.?

Ex. 157. An ice-boat is sailing north in a $10\frac{m}{h}$ west wind, with sail set S. 30° E. Assuming that the resistance of the ice is zero, what is the limit of the velocity of the boat?

Ex. 158. *A* is going west at the rate of 3 miles per hour, and *B* is going north at the rate of 2 miles per hour. *B* is 4 miles directly north of *A* at 2 o'clock. How fast are they separating at 4 o'clock, and at what time do they begin to separate?

Solution: $x=3t-6;\ y=2t;\ v_x=3;\ v_y=2;\ x_{(t=4)}=6;\ y_{(t=4)}=8;$

$$r_{(t=4)}=10;\ \frac{dr}{dt}=v_x\left(\frac{x}{r}\right)+v_y\left(\frac{y}{r}\right)=1.8+1.6=3.4.$$

Hence they separate at the rate of 3.4 miles per hour at 4 o'clock. For the minimum value of r we have

$$\frac{dr}{dt}=0=v_x\left(\frac{x}{r}\right)+v_y\left(\frac{y}{r}\right),$$

$$\frac{3(3t-6)}{r}=-\frac{2(2t)}{r},$$

$$13t=18,$$

$$t=\tfrac{18}{13}.$$

That is, they begin to separate at $\frac{5}{13}$ of an hour after 1 o'clock.

To solve the problem graphically, project each velocity on the line joining the two points and combine the projections.

Construct the problem, and verify by the figure the value $\frac{dr}{dt}=3.4$.

Ex. 159. Two straight railways intersect at an angle of 60°. An engine approaches the intersection on one of the tracks at the rate of $25\frac{m}{h}$, and on the other track an engine is leaving it at the rate of $30\frac{m}{h}$. At what rate are the engines separating when each is 10 miles from the intersection?

Ex 160. (*a*) What is the acceleration of 10 lb. under a force of 20 poundals?

(*b*) What is the acceleration of 1 kilogram under a force of 1,000,000 dynes?

Ex. 161. A mass weighs 1 oz. and is acted on by a force of 1 g. Find the acceleration.

Ex. 162. (*a*) What is the acceleration of 1 ton under a force of 1000 lb.?

(*b*) What is the acceleration of 1 kilogram under a force of 2000 dynes?

Ex. 163. One poundal acts upon 1 g. What is the acceleration?

Ex. 164. One dyne acts upon 1 lb. What is the acceleration?

Ex. 165. A silver bar is 10 ft. long and its cross-section is 1 in. square. A pressure at one end of the bar produces an acceleration of $10\frac{f}{ss}$ in the direction of its length. What is the stress at the mid-section of the bar?

Take the bar as the axis of x, and construct a line whose ordinate shall give the stress at the corresponding section of the bar.

Ex. 166. A uniform vertical straight rod is given an upward acceleration of $161\frac{f}{ss}$ by a pull at its upper end. What is the stress at the mid-section of the rod?

Ex. 167. A uniform vertical straight rod is given an upward acceleration of $g\,\frac{f}{ss}$ by a pressure on its base. What is the stress at the mid-section of the rod?

Ex. 168. The unit of force is 1 lb._w, the unit of mass is 1 lb._m, and the unit of time is 1 second. What is the corresponding unit of distance?

Ex. 169. The force is given in pounds and the acceleration in miles per hour. What is the corresponding unit of mass?

Ex. 170. Taking the value of g to be 32 when the foot and second are the units of length and time, find its value in terms of the yard and minute as units of length and time.

Ex. 171. Taking the value of g to be 981 when the cm. and second are the units of length and time, find its value in terms of the meter and minute as units of length of time.

Ex. 172. Where $g=32$ in the British System, find its value in a system in which the unit of length is two meters and the unit of time is 6 seconds.

Ex. 173. Express an acceleration of $10\frac{f}{ss}$ in miles per hour.

Ex. 174. A force of 100 poundals acts for 5 seconds on a mass of 12.5 lb., starting from rest. How far does the body move during the fifth second?

Ex. 175. A body falls freely in vacuo. What is the distance described in the tenth second?

Ex. 176. A stone is dropped into a mine 64.4 ft. deep. How long does it take to reach the bottom?

Ex. 177. A balloon is ascending with a velocity of $40\frac{f}{s}$. At the height of 200 ft. a body is dropped from the balloon. How long does it take to reach the ground, and with what velocity does it strike?

Ex. 178. A stone is thrown downward into a mine 64.4 ft. deep with a v of $32.2\frac{f}{s}$. How long will it take the stone to reach the bottom?

Ex. 179. A body is projected downward with a velocity of $161\frac{f}{s}$ from the top of a tower 500 ft. high. How long will it take to reach the ground?

Ex. 180. A body is projected downward with a velocity of $100\frac{f}{s}$. How far does it go in the first second?

Ex. 181. A body is projected upward with a velocity of $161\frac{f}{s}$. How far and how long will it rise?

Ex. 182. A body is projected upward in vacuo with a velocity of $100\frac{f}{s}$. How far does it rise in the first second?

Ex. 183. A body falls freely in vacuo. Find the distance described from $t=3.4$ to $t=7.8$, and the mean velocity over this distance.

Ex. 184. A body projected upward reaches a height of 1000 ft. What is the velocity of projection, and how long does the body rise?

Ex. 185. A body weighs 161 lb., and its v is $100\frac{f}{s}$. It is brought to rest by a constant force in ten seconds. Find the intensity of the force and the length of the path over which it acts.

Ex. 186. Find the value of g at West Point, latitude 41° 23′ 31″, at the sea level.

Ex. 187. At what latitude is $g=32.173\frac{f}{s}$ at the sea level?

Ex. 188. A plane is inclined 30° with the horizontal. What is the acceleration of a body sliding down the plane under the action of its own weight, without friction?

Ex. 189. A plane is inclined 30° with the horizontal. A body on the plane moves under the action of its weight and friction. Weight = 32 lb. Friction = 15 lb. Find the acceleration.

Ex. 190. A body moves down a plane inclined at an angle of 45°, the forces being weight and friction. The acceleration is $10\frac{f}{s}$. What is the coefficient of friction?

Ex. 191. What is the A of a body down an inclined plane, angle = 45°, under the weight of the body and $f=\frac{2}{10}$?

Ex. 192. A plane is inclined at an angle of 60° with the horizontal. A body on the plane weighs 20 lb., and the friction between the body and plane is 5 lb.

Find the acceleration down the plane.

How long will it take the body to move 100 ft. down the plane starting from rest? Assume $g=32$.

Ex. 193. Draw (0,40) (40,70) to represent an inclined plane.

A body weighing 10 lb. is projected up the plane with a velocity of $224\frac{f}{s}$.

Friction = 1 lb.

(*a*) How long will it take the body to come to rest?

(*b*) How far will it move up the plane?

(*c*) How long will it take to come back to the starting point?

Ex. 194. A body weighing 100 lb. is on a plane inclined 60° with the horizontal.

The coefficient of friction is $\frac{1}{10}$.

What constant force, acting parallel to the plane, is necessary to move the body 10 ft. up the plane in 1 second? (From rest.)

Ex. 195. A cord passes over a cylinder. On one end of the cord there is a weight of 20 lb. and on the other a weight of 30 lb. Assume the cord to move on the cylinder without friction, and find the acceleration of the weights and the tension on the cord.

Ex. 196. In 195 take the friction of the cord on the cylinder to be 2 lb., and find the acceleration of the weights and the tension in each vertical part of the cord.

Ex. 197. Take a weight of 2 lb. at (25,20), 4 lb. at (25,30), a cylinder of radius 5 with center at (30,50), a weight of 6 lb. at (35,40) and 8 lb. at (35,30). Connect the weights by a cord running over a cylinder.

Find the tension on each branch of the cord, without friction on the cylinder.

Ex. 198. Find the tension on each branch of the cord in 197 with 4 lb. friction on the cylinder.

Ex. 199. Draw (0,30) (16,30) to represent a horizontal plane.

Draw a circle with center at (18,30) and radius 2, representing a pulley.

Draw (0,30) (0,34) (4,34) 4,30) (0,30) to represent a weight of 16 lb. on the plane.

From (4,32) draw a taut cord over the pulley, and then downward, suspending a weight of 12 lb. on the lower end of the cord.

Assume the pulley and cord to be without mass and friction, the plane to be frictionless, and the scale to be 0.1 in. = 1 ft.

Find the tension on the cord, and the time for the weight to move from rest to contact with the pulley.

Ex. 200. Solve 199, assuming the friction between the weight and the plane to be 4 lb.

Ex. 201. In 199 rotate the plane downward about (18,30) until it makes an angle of 30° with the horizontal. Solve the problem, leaving the other conditions as in 199.

Ex. 202. Solve 201, assuming the friction between the weight and the plane to be 2 lb.

Ex. 203. Assume the same plane as in 202, with the same body on the plane and the same friction.

A cord attached to the body extends parallel to the plane and over a pulley at the top, and a weight of 20 lb. is suspended from the end of the cord.

Omit stiffness and weight of cord, and friction and weight of pulley.

(*a*) Find the acceleration of the system.

(*b*) Find the tension on the cord by considering the forces acting on the body which moves up the plane.

(*c*) To check *b* find the tension on the cord by considering the forces which act on the falling weight.

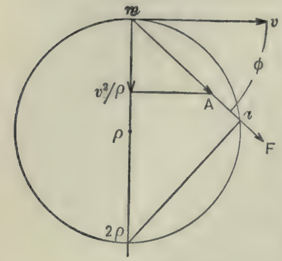

FIG. 41.

Ex. 204. Show that the speed of a particle describing a curved path is equal to that which would be generated by the acting force (considered as constant) in moving the particle from rest over a path equal to one-fourth of the chord of the circle of curvature on the action line of the force. Fig. 41.

The path necessary to generate the velocity from rest is

$$p=\frac{v^2}{2A}, \text{ and } A=\frac{\left(\frac{v^2}{\rho}\right)}{\sin\phi}.$$

The chord of the circle of curvature on the action line of F is $2\rho \sin\phi$, and these relations give $p=\frac{1}{4}ma$.

Ex. 205. A point is revolving about a center under the condition $\theta=10t$. How many revolutions does it make per minute?

Ex. 206. A wheel turns so that $\theta=\pi t^2$. What is the angular velocity? What is the angular acceleration? How many turns per minute does the wheel make when $t=10$? How many turns does the wheel make from $t=0$ until $t=10$?

Ex. 207. An angle is described under the condition $\theta=10t^2$. What is the angular velocity at the time $t=10$? What is the increase in the angular velocity per unit of time?

Ex. 208. An angle is described under the condition $\theta=5t^3$. Find the angular velocity and acceleration at the time $t=5$.

Ex. 209. A wheel turns so that $\theta=2\pi t^3$.

(*a*) Find the angular acceleration.

(*b*) Find the angular velocity when $t=5$.

(*c*) Find the number of turns per minute when $t=5$.

(*d*) How many turns does the wheel make in the first 5 seconds?

Ex. 210. A wheel turns so that $\theta=-t^2$. Find the angular velocity and angular acceleration when $t=10$, and state in what unit they are given.

Ex. 211. A wheel is turning at the rate $\frac{d\theta}{dt}=4\pi$.

How many turns will it make while coming to rest under an angular retardation of $\frac{1}{4}$?

Ex. 212. A point moves so that $x=4t^2$ and $y=t^3$. Construct the velocity and acceleration at the time $t=1$, and the components of the acceleration in the direction of s and ρ.

Ex. 213. A point moves from rest under the conditions

$$x=(\tfrac{3}{2})t^2-5t,\ y=2t^2-9t.$$

Construct the resultant v and A at the time $t=3$, and the tangential and radial components of A.

Ex. 214. E's v is (20,60) (8,60), and its acceleration is (20,60) (32.44). Construct $\frac{v^2}{\rho}$ and $\frac{d^2s}{dt^2}$. Find ρ. Find $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$, about the center of curvature.

Ex. 215. Let a point be moving under the following conditions:

$$x=8t+5,$$
$$y=6t+10.$$

Find the velocity and the acceleration. Is the motion uniform, or varied? Rectilinear, or curvilinear?

Ex. 216. Let a point be moving under the following conditions:

$$x=8t^2+5,$$
$$y=6t^2+10.$$

Find the velocity and the acceleration. Is the motion uniform, or varied? Rectilinear, or curvilinear?

Ex. 217. Let a point be moving under the following conditions:

$$x=8t^3+5,$$
$$y=6t^3+10.$$

Find the velocity and the acceleration. Is the motion uniform, or varied? Rectilinear, or curvilinear?

Ex. 218. Let a point be moving under the following conditions:

$$x=10t,$$
$$y=5t^2.$$

Construct the path from $t=0$ to $t=2$.

Construct the velocity and the acceleration at the time $t=2$.

Construct the $\frac{d^2s}{dt^2}$ and $\frac{v^2}{\rho}$ components of the acceleration at the time $t=2$, and find the corresponding value of ρ.

Construct the circle of curvature at the origin.

Ex. 219. An engine is going around a railroad curve at the rate of $30\frac{m}{h}$. Its acceleration toward the center is $2\frac{f}{s}$.

What is the radius of the curve?

What is the angular velocity of the train?

Ex. 220. $v=24$, $\rho=48$, $\frac{d^2s}{dt^2}=16$.

Construct the acceleration.

What is its numerical value?

What angle does the acceleration make with the path?

Ex. 221. Venus revolves about the sun at a distance of 67,200,000 miles in a practically circular orbit, and in a period of 225 days. Find her acceleration under these conditions.

Ex. 222. $x=15t+30$, $z=10t^2+30$. Construct v_x, v_z, A, A_t, A_ρ, and ρ, for $t=1$.

Ex. 223. Given $\frac{ds}{dt}=8$, $\frac{d^2s}{dt^2}=-3$, $\rho=16$.

Find $\frac{dv}{dt}$, $\frac{d\theta}{dt}$, $\frac{d^2\theta}{dt^2}$.

Construct the problem.

Ex. 224. $\frac{d^2x}{dt^2}=16$, $\frac{d^2y}{dt^2}=12$, $\frac{dx}{dt}=12$, $\frac{dy}{dt}=16$. Find the radius of curvature of the path, and indicate it on the diagram.

Ex. 225. $x=12t$, $z=32t-8t^2+50$.

Construct v and ρ for $t=1$, and find the angular velocity and angular acceleration about the center of curvature.

Ex. 226. A wheel turns from rest under the condition $\theta=3t^3$.

(*a*) What is its angular velocity when $t=5$?

(*b*) How many complete turns does it make from $l=5$ until $t=10$.

Ex. 227. A point starts from (0,0) with $v=(0,0)\ (16,32)$, $A_x=0$, $A_z=-32$.

Plot the point for each quarter second up to 2 seconds, and trace the path.

Draw the circle of curvature at (16,16).

Ex. 228. A point starts from (0,0) with velocity (0,0) (8,24), $\frac{d^2x}{dt^2}=0$, $\frac{d^2z}{dt^2}=-8$.

(*a*) Construct the path described from $t=0$ to $t=6$.

(*b*) Construct the circle of curvature for $t=3$.

(*c*) Find the angular velocity about the center of curvature at $t=3$.

(*d*) Find the angular acceleration about the center of curvature at $t=3$.

(*e*) Find the angular velocity about the origin at $t=3$.

(*f*) Find the angular acceleration about the origin at $t=3$.

(Numerical values required for *e* and *f* to be taken from the figure by scale.)

Ex. 229. Deduce the equation of the trajectory in vacuo by taking any point of the path to be $\frac{1}{2}gt^2$ directly below the point which would be reached with v_i without g.

Ex. 230. Show that for $v_i=0$ the distance from the origin is $x=a\cos\left(\frac{2\pi t}{\tau}\right)$.

Ex. 231. What would be the orbit of a particle under the action of a repulsion varying directly as the distance from a fixed point?

Ex. 232. A cord is fastened to a fixed center and has a weight attached to the free end. The cord is held taut and horizontal, and then the weight is allowed to fall from rest. The strength of the cord is equal to the weight. At what angle will it break?

Ex. 233. A steel rod 4 ft. long makes 1200 revolutions per minute about an axis through its middle and perpendicular to its length. Take the rod as the axis of x, and the rotation axis as the axis of y. Let y represent the stress at x. Deduce the equation of the curve of stress, and construct it.

Ex. 234. A thin steel ring 2 ft. in diameter has an elastic limit of 120,000 lb. per sq. in., and its modulus of elasticity is 30,000,000 lb. per sq. in. What is the greatest number of revolutions per minute the ring can make without receiving a permanent set?

Ex. 235. Motion of a particle on an inclined plane under the action of gravity, without friction.

Let i be the angle between the plane and the horizontal. Take the axes of x and z in the plane, Z making the angle i with the horizontal. Then we may write,

$$\frac{d^2x}{dt^2}=\frac{W\cos\alpha}{m}+\frac{N\cos\theta_x}{m}=0,$$

$$\frac{d^2y}{dt^2}=\frac{W\cos\beta}{m}+\frac{N\cos\theta_y}{m}=0,$$

$$\frac{d^2z}{dt^2}=\frac{W\cos\gamma}{m}+\frac{N\cos\theta_z}{m}=g\sin i.$$

The problem of motion is, therefore, similar to that of a particle in a vertical plane under the acceleration g, modified by the smaller value of the acceleration and by the change in direction.

The velocity at any point is a function of z only. That is, the particle has the same velocity for a given value of z as if it had reached that value of z by any other path. For example, starting from rest, the velocity for free fall is $\sqrt{2gz}$. This value of z would be be reached on the plane over a path $\frac{z}{\sin i}$, with the acceleration $g \sin i$. Its velocity would therefore be

$$\sqrt{2(g \sin i)\left(\frac{z}{\sin i}\right)} = \sqrt{2gz}.$$

Ex. 236. Assume a vertical circle and a series of chords meeting at the highest (or lowest) point. Show that the time of fall down these chords, considered as paths of frictionless constraint, is a constant.

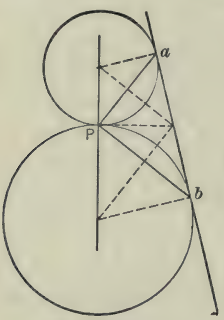

FIG. 42.

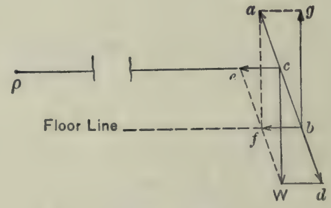

FIG. 43.

Ex. 237. Find the line of quickest descent by frictionless constraint from a given point to a given straight line, both point and line being in the same vertical plane.

Find the line of quickest descent from the line to the point.

The solutions are given in Fig. 42, P being the point and ab the line. Pb is the line of quickest descent from the point to the line, and aP from the line to the point. Explain why.

Ex. 238. Find the general value of $\frac{mv^2}{\rho}$ for frictionless constraint under the action of gravity.

$R_s = R = -mg$, and $\frac{d^2z}{dt^2} = -mg$. Integrating we thus have

$$v^2 = -2gz + C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let v_i and z_i be the initial values of v and z. Then $C=v_i^2+2gz_i$, and (1) becomes

$$v^2=v_i^2+2g(z_i-z). \quad \ldots\ldots \quad (2)$$

Therefore

$$\frac{mv^2}{\rho}=\frac{mv_i^2}{\rho}+\frac{2mg(z_i-z)}{\rho}. \quad \ldots\ldots \quad (3)$$

Ex. 239. Find the least velocity at the highest point of a vertical circle in order that a body of mass m may have continuous motion on the concave side of the arc under the action of gravity only.

The limit of N the normal reaction at the highest point is zero, and

$$N=\frac{mv_i^2}{\rho}-mg=0, \quad \ldots\ldots \quad (1)$$

$$v_i=\sqrt{\rho g}. \quad \ldots\ldots \quad (2)$$

Ex. 240. Find the velocity and normal reaction at the lowest point under the conditions given in Ex. 239. From (2) Ex. 239, and (1) Ex. 239, we have

$$v=\sqrt{5\rho g}, \quad \ldots\ldots \quad (1)$$

$$N=6W. \quad \ldots\ldots \quad (2)$$

Ex. 241. A man swings through an arc of 180° on the flying rings, with arms outstretched. Find the maximum stress on his wrists, in terms of his weight.

How can he swing between the same limits and relieve a part of this stress?

Ex. 242. A man is running on a horizontal floor at the rate of 100 yd. in 15 seconds, and the coefficient of friction between his feet and the floor is $33\frac{1}{3}$ per cent.

(*a*) What is the shortest radius he can turn under these conditions?

The friction on the floor may be considered as producing the deflecting force, and therefore we have $\frac{mv^2}{\rho}=\frac{1}{3}W=\frac{1}{3}mg$. Solving for the radius we have $\rho=37.5$ ft., nearly.

(*b*) At what angle is his body inclined while he is making the turn?

The body must be inclined inward until the horizontal component of the weight is equal to $mv^2\rho$, the other component being in the line of inclination of the body. Thus, in Fig. 43, let ad be the line of inclination of the body, the center of mass being at c. The floor line is indicated on fb, and $c\rho$ is the radius of curvature of the path of c. The horizontal reaction of the floor is $bf=\frac{mv^2}{\rho}$, and the vertical reaction is $bg=-W$. The resultant reaction is ba, equal to the cd component of W, while $ce=\frac{mv^2}{\rho}$ is the horizontal component of W. Thus cd and ba are in equilibrium, leaving ce to deflect the body into the arc whose radius is ρ. Under the given conditions the inclination of the body from the vertical is $\tan^{-1}\frac{1}{3}$.

When a running track turns a corner the floor should be inclined so as to be nearly normal to the inclination of the body for the speed at which the corner is to be turned.

Ex. 243. Compute the elevation of the outer rail (standard gauge = 56.5 in.) for a speed of 30 miles per hour on a curve of 1000 ft. radius, in order that the thrust on the rails shall be normal to the road-bed.

Ex. 244. In making the giant's swing what is the maximum stress on the wrists, provided the center of gravity describes the arc of a circle and the velocity is not retarded by muscular effort, assuming zero velocity at the highest point?

Ex. 245. A particle, under the action of gravity, moves from rest at the highest point of a sphere. Where will it leave the surface?

(Strictly the particle could not move from rest in this case, since it is in equilibrium at the highest point. In this and in the preceding example the initial velocity (at the highest point) may be taken so small as to be negligible in its effect on the velocity to be found in the solution. That is v^2 may be taken equal to $2gh$, within any assignable error.)

At the point where the particle leaves the sphere we have $\frac{mv^2}{\rho}=mg\cos\phi$. Hence

$$\frac{2g(\rho-\rho\cos\phi)}{\rho}=g\cos\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\phi=\cos^{-1}\tfrac{2}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Ex. 246. Find the value of $\frac{v^2}{\rho}$ at the equator, due to the rotation of the earth.

$$\frac{v^2}{\rho}=\frac{\omega^2\rho^2}{\rho}=\omega^2\rho=\frac{(4\pi^2)(5280)(3963.3)}{(86{,}400)^2}. \quad . \quad . \quad . \quad . \quad (1)$$

This acceleration is about $\frac{1}{290}$ of the value of g at the equator, and hence if the earth should rotate seventeen times as fast as it now does, nearly all of the attraction of the earth would be required to deflect a body on the equator into its circular path about the axis of the earth. That is, the apparent weight of the body would be nearly zero.

Ex. 247. A cycloid has its axis of symmetry vertical and is convex upward. A particle moves from rest at the highest point, under the action of gravity. Where will it leave the curve?

Take the origin at the highest point, Z vertical and positive downward. Use the equation $l^2=4az$, l being the length of the curve from the origin to the point whose ordinate is z, and a the diameter of the generating circle.

Assume $\rho=2\sqrt{a^2-az}$.

Ex. 248. A cylcoid has its axis of symmetry vertical and is concave upward. A particle moves on the concave side under the action of gravity.

(*a*) Find the normal reaction at the lowest point when the fall is from rest at the highest point. (See Ex. 247.)

$$N=\left(\frac{mv^2}{\rho}\right)-R\cos\phi=\left[\frac{2mg(a-z)}{\rho}\right]-\left[mg\sqrt{\frac{1-z}{a}}\right]=2mg \text{ for } z=0. \quad (1)$$

(*b*) Find the time of fall from rest over any portion of the curve, from the height h to z.

We have

$$\left(\frac{dl}{dt}\right)^2=2g(h-z), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and from the equation of the curve we get

$$\left(\frac{dl}{dt}\right)^2=\left(\frac{a}{z}\right)\left(\frac{dz}{dt}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equating the second members of (2) and (3) and solving for dt

$$dt=\sqrt{\frac{\frac{1}{2}a}{g}}\,\frac{dz}{\sqrt{hz-z^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence the time of fall from h to z is

$$t=\sqrt{\frac{\frac{1}{2}a}{g}}\int_h^z\frac{dz}{\sqrt{hz-z^2}}=\sqrt{\frac{\frac{1}{2}a}{g}}\operatorname{versin}^{-1}\left(\frac{2z}{h}\right)_h^z. \quad . \quad . \quad (5)$$

For the time of fall from h to $z=0$ this integral between limits gives $t=\pi\sqrt{\frac{\frac{1}{2}a}{g}}$, which is independent of h. We therefore conclude that the time of fall from any point to the lowest point is a constant.

Ex. 249. Find the time of fall down a radius of curvature of the cycloid.

Let α be the angle which ρ makes with the horizontal. Then for the time of fall down ρ we have

$$t=\sqrt{\frac{2\rho}{g\sin\alpha}}=\sqrt{\frac{2\rho}{\left[g\sqrt{1-\left(\frac{dz}{dl}\right)^2}\right]}}$$

$$=\sqrt{\frac{2\rho}{\left[g\sqrt{\frac{1-z}{a}}\right]}}=\sqrt{\frac{4a\sqrt{1-\frac{z}{a}}}{\left[g\sqrt{1-\frac{z}{a}}\right]}}=2\sqrt{\frac{a}{g}}. \quad . \quad (1)$$

which is independent of ρ, and, therefore, a constant for all radii of curvature.

Ex. 249. (*a*) Show that the rotation of the earth causes the action-line of the apparent weight of a body to differ from that of the attraction of the earth for the body.

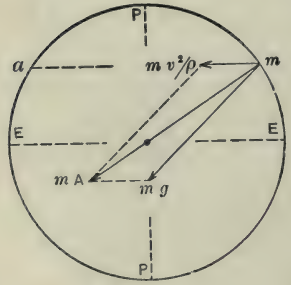

FIG. 44.

In Fig. 44 let PP be the line of the poles, EE the equator, and m a mass on the circle of latitude am.

The acceleration of m toward the center of its circle of latitude is $\frac{v^2}{\rho}$, and this acceleration is produced by the $\frac{mv^2}{\rho}$ component of the attraction of the earth, mA, assumed to act toward the center of the earth. The other component of mA is mg, the *apparent weight.* The apparent weight, therefore, is not directed toward the center of the earth, except at the poles and on the equator.

Ex. 250. Find the work done in lifting a body weighing 100 lb. through a height of 10 ft. Represent the work graphically.

Ex. 251. A body is moved up a smooth inclined plane. Show that the work done is the same as if the body were lifted vertically through an equal height, no matter what may be the direction of the force in moving the body up the plane.

Ex. 252. A force varies directly as the path, and its intensity is 10 lb. when the path is 5 ft. Find the work done over a path of 20 ft. (The intensity is 2 when the path is 1. Hence $F=2p$, and $\int_0^p Fdp=\int_0^{20} 2pdp=20^2=400$ ft. lb.)

Ex. 253. A chain 20 ft. long weighs 5 lb. per foot, and it is suspended vertically from one end. How much work is required to lift the free end 40 ft., the other end remaining fixed?

(After solving $\int_0^{40} Fdp$, check by the product of the mean weight by the height, and also by the product of the weight of the chain by the vertical path of its center of gravity.)

Ex. 254. For a certain spiral spring $F=(2p)$ lb., p in inches.

Find the work done between $p=2$ and $p=4$, in foot-pounds, and represent the work graphically.

Ex. 255. A spring balance is elongated 1 in. by a force of 5 lb. Find the work done in elongating the spring 4 in.

Ex. 256. A force of 2 lb. elongates a spring 1 ft. Represent the work graphically.

Ex. 257. A force of 2 lb. elongates a spring 1 in. Find the work done in elongating the spring 1 ft. Represent the work graphically.

Ex. 258. It takes 600 lb. to compress a spiral spring 1 ft. How much work is done in compressing it 2 ft.?

Ex. 259. A foot cube of silver rests on a horizontal plane. How much work is required to overturn it about one edge?

$$\text{Work}=\int_{45^\circ}^{90^\circ} W\cos\theta\sqrt{\tfrac{1}{2}}\,d\theta,$$

in which θ is the angle between the diagonal of a vertical face and the horizontal plane.

Make a diagram to explain the above integral.

Check by product of weight and vertical path of the center of gravity.

Ex. 260. A chain 20 ft. long weighs 5 lb. per foot. Its upper end is attached to the lowest element of the drum of a windlass, the chain being suspended vertically. The drum is 1 ft. in diameter. How much work is required to wind the entire chain on the drum, considering the weight only?

Ex. 261. The heat potential of 1 lb. of anthracite coal is 3,500,000 therms (approximately). What is the work equivalent of a ton of coal in horse-power for one hour?

Ex. 262. How much bituminous coal (heat potential 3,700,000 therms per lb.) would be required to run a 10 horse-power engine for eight hours, provided only 10 per cent of the heat potential of the coal be utilized?

Ex. 263. A body weighing 20 lb. falls from rest through a height of 10 ft.

What is the work done?

Find the velocity by equating the work to $\frac{mv^2}{2}$.

Ex. 264. A mass of 10 lb. falls freely 100 ft. from rest.

(*a*) Find v by equating the work done to the kinetic energy acquired.

(*b*) Find v directly.

Ex. 265. A stone weighing 6 kilogrammes falls from rest. What will be its kinetic energy at the end of ten seconds?

Ex. 266. A projectile weighing 10 lb. is fired vertically upward with a velocity of $1288\frac{f}{s}$. What is its potential energy with respect to a horizontal plane at the origin when it has attained its maximum height?

Ex. 267. A mass of 1.1 lb. hanging vertically draws a mass of 15 lb. along a frictionless horizontal plane. What is the kinetic energy of the system after moving ten seconds from rest?

Ex. 268. A shot weighing 1000 lb. and having a velocity of $2000\frac{f}{s}$ penetrates 25 ft. into a parapet. What is the mean resistance?

Ex. 269. An ounce bullet penetrates a stick of timber and comes to rest in $\frac{1}{1000}$ of a second. The striking v is $1000\frac{f}{s}$. What is the mean resistance of the stick?

Ex. 270. A projectile striking with $v=3000\frac{f}{s}$ penetrates 12 in. of armor plate. Assuming that the resistance offered by the plate is uniform, what thickness will it penetrate with $v=1200\frac{f}{s}$?

Ex. 271. A car weighing 40 tons is traveling at the rate of 40 $\frac{m}{h}$ when it meets with a constant resistance that stops it in 10 yd. What is the intensity of the resistance, and what is the amount of heat produced, assuming that all the energy of the car is converted into heat?

Ex. 272. A 1-lb. projectile traveling horizontally with $v=800\frac{f}{s}$ strikes an iron plate weighing 500 lb. and suspended by a wire 20 ft. long. If all the energy of the projectile were transferred to the plate through what angle would the plate swing?

Ex. 273. The flow of a river is 28,000 cu. ft. per second, with a fall of 400 ft. Find the horse-power developed.

Ex. 274. A baseball weighs 5 oz. and moves with $v=80\frac{f}{s}$. If caught with a constant pressure of 100 lb. what is the path of the hands in bringing the ball to rest?

Ex. 275. A power pump is making 60 strokes per minute and delivers water 88 ft. above the source—1 cu. ft. of water per stroke. The effective work is $\frac{2}{3}$ the rated horse-power of the engine. Find the horse-power of the engine.

Ex. 276. A weight of 100 lb. falls from rest through a height of 10 ft., and then strikes a spiral spring, compressing it 1 ft. What is the resistance of the spring for this compression?

Ex. 277. The water from a full tank 20 ft. deep and 20 ft. in diameter is pumped to the height of 90 ft. above the top of the tank and delivered with a v due to a height of 50 ft. What is the work done?

Ex. 278. A circular saw $5\frac{1}{2}$ ft. in diameter makes 5 turns per second while sawing with 100 lb. resistance on its edge. Find the horse-power consumed by the saw.

Ex. 279. A 16-lb. shot falls 10 ft. from rest. It then strikes a spring which brings it to rest in a further fall of 2 ft. What is the tension of the spring when the shot comes to rest?

Ex. 280. A free body whose mass is g lb. moves from rest under the action of a force which varies directly as the time, its intensity being 6 lb. at the end of the first second.

Find the work done in ten sconds.

Ex. 282. A spring balance is elongated 1 in. by a 6-lb. pull. A 30-lb. weight is attached to the hook and allowed to fall when the index is at the zero of the scale. How far will the weight fall? Find the work done over this distance. Represent first, the work of the weight; second, the work of the spring.

Ex. 283. A weight of 100 lb. falls 8 ft. from rest, and then strikes a spring whose resistance is 600 lb. at 1 ft. compression. What is the horse-power of the spring when the falling weight has compressed it 1 ft.?

Ex. 284. A spiral spring is stretched 1 ft. by a force of 1 lb.

A 2-lb. weight is held at rest on the hook of the suspended spring when the index is at zero. How far will the weight fall before coming to rest under the action of the spring? Make a diagram to show graphically the work of the spring and the work of the weight during this fall, and the relation between these two quantities of work. Find the horse-power developed on the falling mass when the path $=1$. Find the kinetic energy of the falling mass when the path $=1$.

Ex. 285. A body weighs g lb. and moves so that $x=\frac{3}{2}t^2$ and $y=2t^2$. Construct the path described in two seconds. Represent graphically the work done in the directions of x, y, and the path, W_x, W_y, and W_s. Show by the figure that $W_x+W_y=W_s$.

Write the value of s as a function of t_0.

(Take unity $=\frac{1}{2}$ in.)

Ex. 286. Take the spring of Ex. 284, place a 10-lb. weight on the hook, and let the weight fall when the index is at the zero of the scale.

With what velocity will the index pass the 10-lb. mark?

How far will the weight fall?

Ex. 287. A body whose mass is 64 lb. moves so that

$$x=12t,$$

$$z=32t-8t^2+96.$$

(*a*) What is the intensity and direction of the force acting?
(*b*) What is the kinetic energy at the time $t=0$?
(*c*) What is the kinetic energy at the time $t=-2$?
(*d*) What is the work done in the first two seconds?
(*e*) What is the path of the body?
(*f*) When does the body cross X?
(*g*) What is the horse-power at the time $t=-2$?
(*h*) What is the horse-power at the time $t=5$?

Ex. 288. C. G. S. units.

$$M=10,$$

$$R_x=10x,$$

$$R_y=R_z=0.$$

$$v=0 \text{ when } t=0.$$

Find the work done between the limits $x=0$ and $x=10$.

From the work done find the velocity when $x=10$.

Check the velocity by finding $\frac{dx}{dt}$.

Ex. 289. **Solution of the General Problem of the Screw by Equating the Work of the Applied Force to the Work of the Load and Friction.** We have applied the principles of equilibrium in the solution of mechanical problems involving the relations between resistances and applied forces, as in the wedge and the screw. In some cases, however, it is convenient to equate the work of the applied force to that of the load and the resistances, assuming uniform motion of the machine. Then by canceling a common path factor the relation between the applied force and the load may be obtained.

For example, take the usual form of screw with triangular thread, as in Fig. 45. The helicoidal thread is generated by the triangle *abc*; and we may assume—as a sufficient approximation—that the friction and the normal pressure on the thread are concentrated on a mean helix, and thus avoid consideration of the variable slope of the helical elements of the thread.

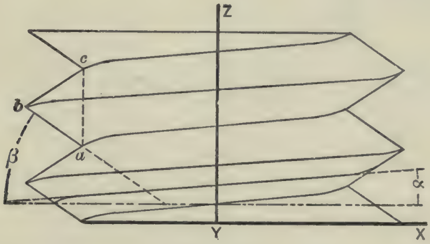

Fig. 45.

Assume the following notation:

α, The inclination of the mean helix with a plane normal to the axis of the screw;
r, The distance of the mean helix from the axis of the screw;
β, The angle between the generating line of the bearing surface, ab, and a plane normal to the axis of the screw;
γ, The angle between the normal to the bearing surface, at the mean helix, and the axis of the screw;
ϕ, The angle through which the screw turns in lifting the load;
l, The lever-arm of F, applied normal to the axis;
W, The load;
N, The normal pressure on the thread, assumed to be concentrated at the mean helix.

Then we have for the equation of work,

$$Fl\phi = \frac{Wr\phi \tan \alpha + fNr\phi}{\cos \alpha}; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or, dividing by ϕ,

$$Fl = \frac{Wr \tan \alpha + fNr}{\cos \alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To eliminate N from this equation we have from the equilibrium in the direction of the axis,

$$W + fN \sin \alpha = N \cos \gamma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

from which

$$N = \frac{W}{(\cos \gamma - f \sin \alpha)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting this in (2) and solving for $\frac{F}{W}$ we have

$$\frac{F}{W} = \left(\frac{r \tan \alpha}{l}\right)\left(\frac{1+f}{[\sin \alpha(\cos \gamma - f \sin \alpha)]}\right). \quad . \quad . \quad . \quad (5)$$

But we have also *

$$\cos \gamma = \frac{1}{(1+\tan^2 \alpha + \tan^2 \beta)^{1/2}}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and this substituted in (5) gives, after reduction,

$$\frac{F}{W} = \left(\frac{r \tan \alpha}{l}\right)\left(1 + \frac{f(1+\tan^2 \alpha + \tan^2 \beta)^{1/2}}{\sin \alpha - f \sin^2 \alpha (1+\tan^2 \alpha + \tan^2 \beta)^{1/2}}\right). \quad (7)$$

For the square thread $\beta = 0$, and this in (7) shows a smaller ratio of F to W for the square than for the triangular thread of the same pitch and diameter.

The limiting ratio, obtained by making $f = 0$, is

$$\frac{F}{W} = \frac{r \tan \alpha}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

as in Ex. 114.

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* The direction cosines of the normal to a surface are of the form

$$\cos \theta_z = \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}.$$

Take the axis of the screw as Z and the foot of the normal in YZ. See Granville's Calculus, Art. 135.

Ex. 290. Find the velocity from infinity at the poles of the earth under the earth's attraction.

Assume the mass of the earth to be 1.3153×10^{25} lbs., the attraction of 1 lb._m for 1 lb._m at the distance of 1 ft. to be $\frac{1.0665}{10^9}$ poundals, and $r=3950$ miles.

Ans. 6.95 miles per second, nearly.

(The method of finding μ is explained in connection with the torsion pendulum.)

Ex. 291. Check the velocity given in Ex. 290 by assuming $g=32.255\frac{f}{ss}$ at the poles, and integrating between ∞ and r under the law of the inverse square.

Ex. 292. Equate the potential of the earth at the poles to the work between infinity and r, and find the mass of the earth in terms of μ. *Ans.* $M=\frac{(3950 \times 5280)(6.726 \times 10^8 \text{ ft.pdls.})}{\mu}$.

Ex. 293. Find the attraction between the sun and the earth at the distance of 93,000,000 miles, and the acceleration of the earth under this attraction.

Assume the mass of the sun to be 332,000 times the mass of the earth.

Ex. 294. Find the surface gravity, potential, and velocity from infinity, as indicated in the following table:

Name of Body.	Given.			Find.		
	Mass.	Mean Diameter.	Mean Distance from the Sun, in Millions of Miles.	Surface Gravity.	Potential at the Surface of the Body under its Own Attraction (Mile-tons).	Velocity from ∞ at the Surface of the Body under its Own Attraction (Miles per Second).
Sun......	332,000	109.4	....	27.65		
Moon....	$\frac{1}{81.5}$	0.273	....	0.1667		
Mercury..	$\frac{1}{21}$	0.382	36.0	0.43		
Venus....	0.82	0.972	67.2	0.82		
Earth....	1	1	92.9	1		
Mars....	$\frac{1}{9.32}$	0.534	141.5	0.38		
Jupiter...	317.7	10.92	483.3	2.65		
Saturn...	94.8	9.17	886.0	1.18		
Uranus...	14.6	4.03	1781.9	0.90		
Neptune .	17.0	4.39	2791.6	0.89		

Ex. 295. Find the Σmr^2 and k^2 of a uniform straight rod of constant density, with respect to a perpendicular axis at one end of the rod.

Let the cross-section of the rod be c, taken so small that all of its points may be assumed to have the same value of r. Take the rod to be of length l on the axis of x, with the axis of Σmr^2 in YZ. Then we have $dM = \delta c dx$ and $M = \delta cl$, and

$$\Sigma mr^2 = \delta c \int_0^l x^2 dx = \tfrac{1}{3}\delta c l^3 = Mk^2, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$k^2 = \frac{l^2}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Ex. 296. Find the Σmr^2 and k^2 of a thin homogeneous rectangular plate with respect to axes through its center, X and Y parallel to its sides and Z normal to its plane.

Let the sides be $2a$ and $2b$. Then, since the plate may be considered as made up of rods parallel to a side, we have by example 295

$$\Sigma mr_x^2 = \frac{Mb^2}{3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\Sigma mr_y^2 = \frac{Ma^2}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For the z axis

$$\Sigma mr_z^2 = \Sigma m(r_x^2 + r_y^2) = \Sigma mr_x^2 + \Sigma mr_y^2 = \frac{M(b^2+a^2)}{3}. \quad . \quad . \quad (3)$$

The corresponding values of k^2 are

$$k_x^2 = \frac{b^2}{3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$k_y^2 = \frac{a^2}{3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$k_z^2 = \frac{(b^2+a^2)}{3}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

that is, k_x is the same as for the y side, k_y the same as for the x side, and k_z the same as for the diagonal.

The value $\Sigma mr_z^2 = \Sigma mr_x^2 + \Sigma mr_y^2$ is general for any plane figure with respect to an axis normal to its plane. That is, *the* Σmr^2 *of a plane figure with respect to any axis normal to its plane is equal to the sum of the* (Σmr^2)*'s with respect to any two rectangular axes in the plane of the figure and intersecting the normal axis.*

Ex. 297. Find the (Σmr^2)'s of a homogeneous rectangular parallelopiped with respect to axes through the center and normal to the faces.

Let a, b, c, be the half edges in the directions of x, y, z, respectively. We may consider the parallelopiped to be made up of plates equal and parallel to any face, and we have, therefore, according to Eq. (6), *Ex.* 296,

$$Mk_x{}^2=\frac{M(b^2+c^2)}{3}, \quad \dots\dots\dots \quad (1)$$

$$Mk_y{}^2=\frac{M(a^2+c^2)}{3}, \quad \dots\dots\dots \quad (2)$$

$$Mk_z{}^2=\frac{M(a^2+b^2)}{3}. \quad \dots\dots\dots \quad (3)$$

That is, the radius of gyration for each axis is the same as for the diagonal of the perpendicular face.

Ex. 298. Find the Σmr^2 and k^2 of a uniform straight rod with respect to an inclined axis at one end of the rod.

Let θ be the angle between the rod and the axis, and $r=x\sin\theta$. Solve as in Ex. 295.

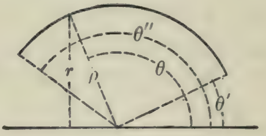

FIG. 46.

Ex. 299. Find the (Σmr^2)'s of a uniform circular arc of constant density, with respect to axes through its center, in and perpendicular to its plane.

1. *Axis in the plane.* Take the arc to be $\rho(\theta''-\theta')$, as in Fig. 46, and indefinitely small cross-section, c. Then we have $dM=\delta c\rho d\theta$ and $r=\rho\sin\theta$, and by integrating between limits,

$$\Sigma mr^2=\delta c\rho^3\int_{\theta}^{\theta''}\sin^2\theta d\theta$$

$$=\tfrac{1}{2}\delta c\rho^3[(\theta''-\sin\theta''\cos\theta'')-(\theta'-\sin\theta'\cos\theta')]. \quad . \quad (1)$$

For the entire circumference $\theta'=0$ and $\theta''=2\pi$, and

$$\Sigma mr^2=\delta c\rho^3\pi=\frac{M\rho^2}{2}, \quad \dots\dots\dots \quad (2)$$

$$k^2=\frac{\rho^2}{2}. \quad \dots\dots\dots \quad (3)$$

2. *Axis Normal to the Plane.* If the axis be taken through the center and perpendicular to the plane of the arc we have, obviously,

$$\Sigma mr^2=M\rho^2. \quad \dots\dots\dots \quad (4)$$

Ex. 300. Check the consistency of (2) and (4), Ex. 299, by $\Sigma mr_z^2 = \Sigma mr_x^2 + \Sigma mr_y^2$.

Ex. 301. Find the Σmr^2 of a uniform circular plate with respect to axes through its center, in and normal to its plane.

1. *Axis in the Plane.* Let c be the indefinitely small thickness of the plate, and ρ its radius. Then $dM = \delta c 2\pi r dr$, and by (1), Ex. 299, the Σmr^2 of dM is $\delta c\pi r^3 dr$. Taking X in the plane we then have

$$\Sigma mr_x^2 = \delta c\pi \int_0^\rho r^3 dr = \tfrac{1}{4}\delta c\pi\rho^4 = \tfrac{1}{4}M\rho^2, \tag{1}$$

$$k_x^2 = \tfrac{1}{4}\rho^2. \tag{2}$$

2. *Axis normal to the plane.* Either by $\int_0^\rho \delta c 2\pi r^3 dr$ or by $\Sigma mr_z^2 = \Sigma mr_x^2 + \Sigma mr_y^2$ we have

$$\Sigma mr_z^2 = \tfrac{1}{2}M\rho^2, \tag{3}$$

$$k_z^2 = \tfrac{1}{2}\rho^2. \tag{4}$$

Ex. 302. Show that the Σmr^2 of a body with respect to any axis is equal to the Σmr^2 with respect to a parallel axis through the center of mass plus the product of the mass of the body by the square of the distance between the two axes.

Let Z be the given axis and Z' the parallel axis through the center of mass, X' and Y' being parallel to X and Y.

The distance between Z and Z' is $d = \sqrt{x_c^2 + y_c^2}$, in terms of the x and y coordinates of the center of mass. Then

$$\Sigma mr_z^2 = \Sigma mx^2 + \Sigma my^2, \tag{1}$$

$$x = x_c + x', \tag{2}$$

$$y = y_c + y', \tag{3}$$

$$\begin{aligned}\Sigma mr_z^2 &= \Sigma mx_c^2 + \Sigma mx'^2 + 2x_c\Sigma mx' \\ &\qquad + \Sigma my_c^2 + \Sigma my'^2 + 2y_c\Sigma my' \\ &= Md^2 + \Sigma mr_{z'}^2 + 2x_c\Sigma mx' + 2y_c\Sigma my'\end{aligned} \tag{4}$$

By the *principle of the center of mass* the last two terms of (4) reduce to zero, and, therefore

$$\Sigma mr_z^2 = \Sigma mr_{z'}^2 + Md^2. \tag{5}$$

Ex. 303. Find the Σmr^2 and k^2 of a hollow uniform right cylinder of circular section and indefinitely small thickness, with respect to an axis through the center of one end and perpendicular to the axis of the cylinder.

The mass of a sectional element of the cylinder is $\delta 2\pi\rho c dl$, the radius, thickness and length of the cylinder being ρ, c, and l. Taking x to be the distance of the sectional element from the axis its Σmr^2 is, by (2), Ex. 299, and (5) Ex. 302, $\frac{(\delta 2\pi\rho c dx)\rho^2}{2}+(\delta 2\pi\rho c dx)x^2$. Therefore we have for the entire cylinder,

$$\Sigma mr^2=\int_0^l(\delta\pi\rho^3 c dx+2\delta\pi\rho c x^2 dx)=\frac{M\rho^2}{2}+\frac{Ml^2}{3}. \quad . \quad . \quad (1)$$

$$k^2=\tfrac{1}{2}\rho^2+\tfrac{1}{3}l^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Ex. 304. Find the Σmr^2 and k^2 of a solid homogeneous right cylinder of circular section, with respect to an axis through the center of one end and perpendicular to the axis of the cylinder.

The mass of a sectional element of the cylinder at the distance x from the given axis is $\delta\pi\rho^2 dx$, and the Σmr^2 of this element is, by (1), Ex. 301, and (5), Ex. 302, $\frac{(\delta\pi\rho^2 dx)\rho^2}{4}+(\delta\pi\rho^2 dx)x^2$. Therefore

$$\Sigma mr^2=\int_0^l(\tfrac{1}{4}\delta\pi^4 dx\rho+\delta\pi\rho^2x^2dx)=\frac{M\rho^2}{4}+\frac{Ml^2}{3}, \quad . \quad . \quad (1)$$

$$k^2=\tfrac{1}{4}\rho^2+\tfrac{1}{3}l^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

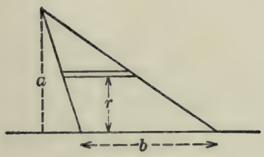

FIG. 47.

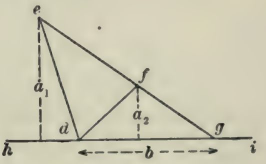

FIG. 48.

Ex. 305. Find the Σmr^2 of a triangle with respect to one of its sides as an axis. Also for any axis in the plane of the triangle and through one vertex.

1. Let b be the length of the side on the axis, assumed as the base of the triangle, a being the altitude, as in Fig. 47. Then, c being the indefinitely small thickness,

$$dM=\delta c\left[\frac{(a-r)}{a}\right]bdr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\Sigma mr^2=\left(\frac{\delta cb}{a}\right)\int_0^a(a-r)r^2dr$$

$$=\frac{\delta cb}{a}\tfrac{1}{12}a^4=\tfrac{1}{6}Ma^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

2. Fig. 48. The triangle is *def* and the axis is *hi*, outside the triangle. Let M be the mass of *def*, M_1, the mass of *deg*, and M_2 the mass of *dfg*. Then

$$M_1 = \tfrac{1}{2}\delta bca_1,$$

$$M_2 = \tfrac{1}{2}\delta bca_2,$$

$$M = M_1 - M_2 = \tfrac{1}{2}\delta bc(a_1 - a_2),$$

$$\Sigma mr^2 = \frac{M_1a_1^2}{6} - \frac{M_2a_2^2}{6} = \tfrac{1}{12}\delta bc(a_1^3 - a_2^3) = \tfrac{1}{6}M(a_1^2 + a_1a_2 + a_2^2). \quad (3)$$

If the axis cut the triangle, we have, solving as for Fig. 48,

$$\Sigma mr^2 = \tfrac{1}{6}M(a_1^2 - a_1a_2 + a_2^2). \quad . \; . \; . \; . \; . \; (4)$$

Ex. 306. Show that the Σmr^2 of a triangle with respect to an axis in its plane and through a vertex is the same as if one-third the mass of the triangle were concentrated at the middle of each side. For Fig. 48,

$$\Sigma mr^2 = \tfrac{1}{3}M(\tfrac{1}{2}a_1{}^2 + \tfrac{1}{3}M(\tfrac{1}{2}a_2)^2 + \tfrac{1}{3}M[\tfrac{1}{2}(a_1 + a_2)]^2$$

$$= \tfrac{1}{6}M(a^{12} + a_1a_2 + a_2^2), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (5)$$

which is the value given in (3), Ex. 305.

Similar proof may be made for Fig. 47 and (4), Ex. 305.

Ex. 307. Show that the Σmr^2 of a triangle with respect to any axis parallel or perpendicular to the plane of the triangle is the same as if one-third of the mass of the triangle were concentrated at the middle of each side.

The thirds at the middle points of the sides have the same center of mass as the triangle, and, therefore, the same Σmr^2 with respect to any axis through the center of mass and parallel to any line for which the equality of (Σmr^2)'s is shown in Ex. 306, by $\Sigma mr_{x'}^2 = \Sigma mr_x^2 - Md^2$. Then the same equality exists for any parallel line in space, by $\Sigma mr_x^2 = \Sigma mr_x^2 + Md^2$.

For an axis normal to the plane of the triangle we have equality of $\Sigma mr_z^2 = \Sigma mr_x^2 + \Sigma mr_y^2$ for the triangle and thirds, since both the (Σmr_x^2)'s and the (Σmr_y^2)'s have been shown to be equal.

(The proposition is true for any axis.)

Ex. 308. Find the Σmr^2 of a homogeneous solid generated by a triangle revolved about one of its sides.

Assume the triangle as in Fig. 47, the axis being the side b.

For the mass of an elementary cylinder we have

$$dM=\delta\left[\frac{(a-r)}{a}\right]b2\pi r dr, \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

whose Σmr^2 is $\delta\left[\frac{(a-r)}{a}\right]b2\pi r^3dr$. The Σmr^2 of the body is therefore

$$\Sigma mr^2=\delta 2\pi b\int_0^a\left[\frac{(a-r)}{a}\right]r^3dr=\delta 2\pi b\left(\frac{\frac{1}{4}a^4-\frac{1}{5}a^5}{a}\right)=\tfrac{1}{10}\delta\pi ba^4=\tfrac{3}{10}Ma^2. \quad (2)$$

($M=\delta\frac{1}{2}ba2\pi\frac{1}{3}a=\frac{1}{3}\delta\pi ba^2$, by theorem of Pappus.)

Ex. 309. Find the Σmr^2 of a thin homogeneous elliptical plate with respect to the axes and the normal at the center of mass.

The equation of the ellipse is $a^2y^2+b^2x^2=a^2b^2$, and for a quadrant of the plate we have (thickness $=c$)

$$\tfrac{1}{4}\Sigma m{r_y}^2\,\delta c\int_0^a yx^2dx=\frac{\delta cb}{a\int_0^a(a^2-x^2)^{1/2}x^2dx}=\tfrac{1}{16}\delta c\pi a^3b, \qquad (1)$$

$$\Sigma m{r_y}^2=\tfrac{1}{4}\delta c\pi a^3b=\tfrac{1}{4}Ma^2. \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

Similarly

$$\Sigma m{r_x}^2=\tfrac{1}{4}Mb^2, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

and by summation

$$\Sigma m{r_z}^2=\tfrac{1}{4}M(a^2+b^2). \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

(The complete integral in (1) is

$$\left(\frac{x}{8}\right)(2x^2-a^2)(a^2-x^2)^{1/2}+\left(\frac{a^4}{8}\right)\sin^{-1}\left(\frac{x}{a}\right)+C,$$

obtained after applying reduction formula [*A*] and then [*B*]. See Granville's Calculus, p. 355, Ex. 8.)

Ex. 310. Find the Σmr^2 of a solid homogeneous ellipsoid with respect to an axis of figure.

The equation of the surface is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. The semi-axes of a section normal to X are $\left(\frac{b}{a}\right)(a^2-x^2)^{1/2}$ and $\left(\frac{c}{a}\right)(a^2-x^2)^{1/2}$, and for a differential slice normal to X we have

$$dM=\delta\pi\left(\frac{bc}{a^2}\right)(a^2-x^2)dx,$$

whose Σmr^2 with respect to X is, by Ex. 309, Eq. (4),

$$\tfrac{1}{4}\delta\pi bc\left[\frac{(b^2+c^2)}{a^4}\right](a^2-x^2)^2dx.$$

Therefore we have

$$\Sigma m r_x^2 = \frac{\frac{1}{4}\delta\pi bc(b^2+c^2)}{a^4\int_a^{-a}(a^2-x^2)^2dx} = \frac{\frac{4}{3}\delta\pi abc(b^2+c^2)}{5} = \frac{M(b^2+c^2)}{5}. \qquad (1)$$

So also

$$\Sigma m r_y^2 = \frac{M(a^2+c^2)}{5}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2$$

$$\Sigma m r_z^2 = \frac{M(a^2+b^2)}{5} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From these we get for a solid spheroid with respect to its axis of revolution, ρ being the radius of the equator,

$$\Sigma m r^2 = \frac{2M\rho^2}{5}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is also applicable to a solid sphere with respect to any diameter.

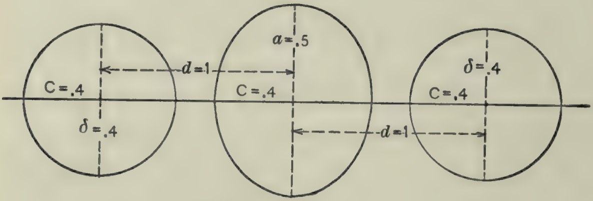

FIG. 49.

Ex. 311. Show that, if the central ellipsoid be an oblate spheroid, the ellipsoid is a sphere at points on the axis of the spheroid at the distance $\sqrt{\dfrac{\left(\dfrac{1}{c^2}-\dfrac{1}{a^2}\right)}{M}}$ from the center of mass. Fig. 49.

The required condition is $\dfrac{1}{\delta^2}=\dfrac{1}{a^2}+Md^2=\dfrac{1}{c^2}$.

Ex. 312. Take the scale of the ellipsoids in Fig. 49 to be 1 in. $=\frac{1}{100}$, and the scale of distance to be 1 in. = 10 ft. Scale the figure and find the mass of the body, the unit of mass being 1 **grav**,* and the unit of distance 1 ft.

---

* A "grav" is the unit of mass when the pound force is the unit of force; i.e., a unit of mass 32.2 times as large as 1 lb. mass, hence to arrive at a proper result in pounds force we divide a mass given in lbs. by 32.2, giving the mass in "*gravs.*"

Ex. 313. Show that, if the central ellipsoid be a prolate spheroid, the ellipsoid is an oblate spheroid at points in the plane of the equator of the central ellipsoid at the distance $\sqrt{\frac{\frac{1}{c^2}-\frac{1}{a^2}}{M}}$ from the center of mass.

Ex. 314. Find the ratio of the length to the radius of a solid homogeneous right cylinder with circular base in order that its central ellipsoid shall be a sphere. *Ans.* $\frac{l}{r}=\sqrt{3}$.

Ex. 315. Find the ratio of the length to the radius of a solid homogeneous right cylinder with circular base in order that its Σmr^2 ellipsoid at the end of the axis shall be a sphere.

Ans. $\frac{l}{r}=\frac{1}{2}\sqrt{3}$.

Ex. 316. Two of the semi-axes of a Σmr^2 ellipsoid are 30 and 40. What is the limit of the other?

Ex. 317. The radius of a rotating cylinder attached to Atwood's machine is 2.54 centimeters, and the mass of the cylinder is 707, M_1 grams. The falling mass M_2 is 15 grams, and the moment of friction, M_f is equal to that of 4 grams with a lever-arm l equal to the radius of the cylinder.

An experiment gives a fall of 160 centimeters in 3.3 seconds. Compare the corresponding acceleration with that given by

$$\frac{d^2\theta}{dt^2}=\frac{(M_2gl-M_f)}{(M_2l^2+M_1K^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Take $g=980\ \frac{\text{cms.}}{ss}$.

Ans. $\frac{d^2z}{dt^2}=29.385$ by experiment; $\frac{d^2\theta}{dt^2}=11.517$ and $\frac{d^2z}{dt^2}=$ 29.254 by Eq. (1), differing from the experimental values by less than $\frac{1}{2}$ of 1 per cent.

Ex. 318. An indefinitely thin cylindrical rim 4 ft. in diameter and weighing 161 lb. rotates about its axis under a belt pull of 40 lb. Find the angular acceleration.

Ex. 319. A fly-wheel is 10 ft. in diameter, its k is 4.5 ft., and it weighs 12,880 lb.

(*a*) How long will it take to get up a speed of 120 turns per minute under a belt pull of 350 lb., resisted by a moment of friction of 130?

(*b*) How many turns will it make while it is getting up this speed?

(*c*) Check by equating the work to the kinetic energy of the wheel.

Ex. 320. A grindstone weighs 1610 lb., is 8 ft. in diameter, and is revolving at the rate of 60 turns per minute. How long will it take it to come to rest under 10 lb. friction on its face?

Solve as a problem of motion, and check by equating the kinetic energy of the stone to the work done in bringing it to rest.

Ex. 321. A fly-wheel weighs 19,320 lbs. and its radius of gyration is 10 ft. An engine with two cylinders runs the fly-wheel by a mean steam pressure of 100 lbs., on piston-heads 18 in. in diameter, with a stroke of 3 ft. How many revolutions does the fly-wheel make in getting up a speed of 120 turns per minute, assuming 10 per cent loss of energy in passing from the cylinders to the fly-wheel?

Ex. 322. Fig. 50. An indefinitely thin hollow right cylinder with circular base rolls down an inclined plane under the action of its weight. Find the acceleration of the axis and the angular acceleration about the axis.

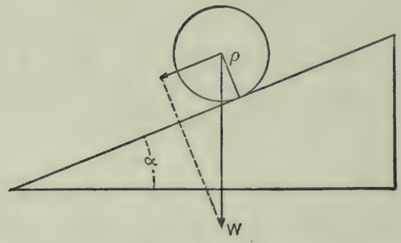

FIG. 50.

Solution:
$$\frac{d^2\theta}{dt^2}=\frac{(Fr)\rho}{(M\rho^2)}=\frac{(Fr)}{(M\rho)},$$

$$\frac{d^2s}{dt^2}=\frac{[(W\sin\alpha-(Fr)]}{M},$$

$$\frac{d^2s}{dt^2}=\frac{\rho d^2\theta}{dt^2}.$$

From which

$$(Fr)=\tfrac{1}{2}W\sin\alpha,$$

$$\frac{d^2s}{dt^2}=\tfrac{1}{2}g\sin\alpha,$$

$$\frac{d^2\theta}{dt^2}=\tfrac{1}{2}\cdot\frac{g\sin\alpha}{\rho}.$$

This example may be solved without considering the friction, as follows:

The angular acceleration about the axis is equal to the angular acceleration about the point of contact. Taking the point of contact as a center we have $W(\rho \sin \alpha)=Mg\ (\rho \sin \alpha)$ for the moment of W, and $\Sigma mr^2=M\rho^2+M\rho^2=2M\rho^2$. Therefore $\frac{d^2\theta}{dt^2}=\frac{Mg\ (\rho \sin \alpha)}{(2M\rho^2)}$ $=\frac{1}{2}\cdot\frac{g \sin \alpha}{\rho}$.

Ex. 323. Check the consistency of the answers to Ex. 322 by $\frac{1}{2}Mv^2+\frac{1}{2}\omega^2\Sigma mr^2=Wh$.

Ex. 324. A solid right cylinder with a circular base rolls down an inclined plane under the action of its weight. Find the acceleration of the axis and the angular acceleration about the axis.

Ex. 325. Check the consistency of the answers to Ex. 324 by $\frac{1}{2}Mv^2+\frac{1}{2}\omega^2\Sigma mr^2=Wh$.

Ex. 326. Fig. 51. A cord is wound around a thin circular hoop and attached to a fixed support, as in Fig. 51. The hoop falls from rest with the cord vertical. Find the acceleration of the center of mass and the angular acceleration about the center of mass, omitting the stiffness and weight of the cord.

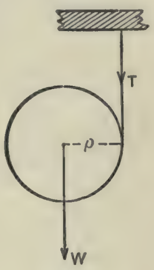

FIG. 51.

Let T be the tension of the cord. Then

$$\frac{d^2z}{dt^2}=\frac{(W-T)}{M},$$

$$\frac{d^2\theta}{dt^2}=\left(\frac{T\rho}{M\rho^2}\right)=\frac{T}{(M\rho)},$$

$$\frac{d^2\theta}{dt^2}=\frac{\left(\frac{d^2z}{dt^2}\right)}{\rho}.$$

From which

$$\frac{d^2t}{dt}=\tfrac{1}{2}g.$$

$$\frac{d^2\theta}{dt^2}=\frac{\frac{1}{2}g}{\rho}.$$

Ex. 327. Test the consistency of the answers to Ex. 326 by the law of the conservation of energy.

Ex. 328. In Ex. 326 replace the hoop by a solid circular disk, and solve for the linear and angular accelerations.

Ex. 329. Test the answers to Ex. 328 by the conservation of energy.

Ex. 330. In Ex. 322 find the limit of α for rolling.

If there be any sliding of the surface of the cylinder on the plane we shall have

$$\frac{\rho d^2\theta}{dt^2} < \frac{d^2s}{dt^2},$$

$$\frac{\rho^2 fW\cos\alpha}{M\rho^2} < \frac{(W\sin\alpha - fW\cos\alpha)}{M},$$

$$2f\cos\alpha < \sin\alpha,$$

and hence there will be sliding if $\tan\alpha > 2f$. (f = coef. friction).

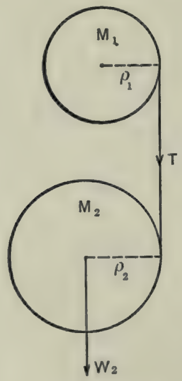

FIG. 52.

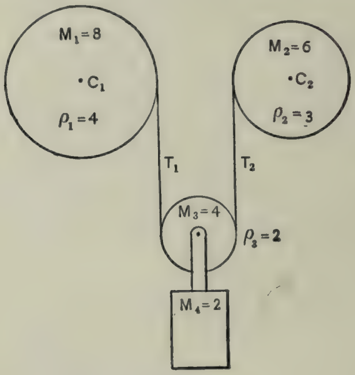

FIG. 53.

The same result is obtained by assuming the value of the friction given by the solution of Ex. 322, and writing $f\,W\cos\alpha = \frac{1}{2}W\sin\alpha$. Beyond this limit of α the friction is not sufficient to prevent sliding, and the extent of the sliding in any case is $(s - \rho\theta)$.

Ex. 331. A homogeneous circular disk is mounted so as to rotate without friction on its axis, fixed in a horizontal position. A cord is passed over the disk, and a mass is suspended from each end of the cord. These masses and that of the disk are 1, 2, and 3, respectively. The friction of the cord on the disk is sufficient to prevent slipping. Solve the problem of motion, neglecting the stiffness and weight of the cord.

Test the answers by the conservation of energy.

Ex. 332. Fig. 52. A homogeneous circular disk of mass M_1 it mounted so as to rotate about a horizontal axis at its center, without friction. Another disk, mass M_2, is arranged so that it can fall under the action of its weight while a cord is wound about both disks, the straight portion of the cord being vertical. Solve for the linear acceleration of the center of M_2, the angular accelerations of the disks, and the tension of the cord.

Neglect the stiffness and weight of the cord.

Test the answers by the conservation of energy.

The four conditions required in the solution are the relations of

$$\frac{d^2\theta_1}{dt^2} \text{ to } T,\ \rho_1, \text{ and } M_1k_1^2,$$

$$\frac{d^2\theta_2}{dt^2} \text{ to } T,\ \rho_2 \text{ and } M_2k_2^2,$$

$$\frac{d^2z}{dt^2} \text{ to } \rho_1,\ \frac{d^2\theta_1}{dt^2},\ \rho_2, \text{ and } \frac{d^2\theta_2}{dt^2},$$

$$\frac{d^2z}{dt^2} \text{ to } W_2,\ T, \text{ and } M_2.$$

Ex. 333. In Ex. 332 assume $M_1=6$, $M_2=8$, $\rho_1=3$, $\rho_2=4$. Solve the problem and check by the conservation of energy.

Ex. 334. Fig. 53. The axes C_1 and C_2 are fixed, and the disks are solid and homogeneous. Neglect friction on journals and the weight and stiffness of the cord.

Solve the problem of motion and check by the conservation of energy.

(The angular accelerations of the disks and the linear acceleration of the falling weights are given directly by the laws of motion. Two additional equations are

$$\frac{\rho_1 d^2\theta_1}{dt^2}=\frac{d^2z}{dt^2}-\frac{\rho_3 d^2\theta_3}{dt^2} \quad \text{and} \quad \frac{\rho_2 d^2\theta_2}{dt^2}=\frac{d^2z}{dt^2}+\frac{\rho_3 d^2\theta_3}{dt^2}.)$$

Ex. 335. The fly-wheel of an engine weighs 42,300 lb. and its radius of gyration is 10 ft., and it is revolving at the rate of 120 turns per minute. A machine working at the rate of 5 horse-power is stopped for one minute. If the fly-wheel should take up the energy which would have been supplied to the machine, without any compensating effect of other machines or by the governor of the engine, what would be the increase in the angular velocity of the fly-wheel?

Ex. 336. Find the Σmr^2 of the fly-wheel of a 250 horse-power engine, so designed that it shall run at 120 turns per minute and take up the work of the engine for one second with a limiting speed of 121 turns per minute.

Ex. 337. A brass cylinder 6 in. in diameter and 8 in. long is suspended by a piano wire in the line of the axis of the cylinder. The cylinder weighs 530 lb. per cubic foot, and a couple consisting of two forces of 70 g. each, applied tangent to the cylinder and in a horizontal plane, deflects it through the unit angle. Find the time of oscillation.

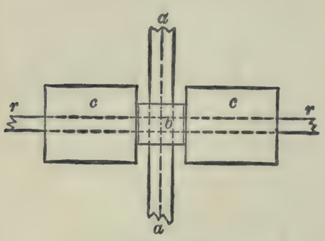

FIG. 54.

Ex. 338. In Fig. 54 *aa* is the axis of a torsion pendulum actuated by a coiled spring surrounding the shaft *aa* and attached to the frame which supports the pendulum. Two equal cylinders, *c,c,* are attached so as to slide on the rod *rr,* to which they may be clamped by set-screws.

The following measurements are given:

Diameter of each cylinder	3.02 cm.
Length of each cylinder.	3.05 cm.
Mass of each cylinder.	176.5 grams
Diameter of *rr*.	0.636 cm.
Diameter of collar on shaft, *b*.	2.077 cm.

A force of 131 grams with a lever-arm of 1.855 cm. deflects the pendulum through the unit angle, and thus measures the torque of the spring. Under the action of this torque the pendulum beats half-seconds when the cylinders are set against the collar on the shaft, *b*, as in the figure. How far must the cylinders be moved out (equal distances) on the rod *rr* in order to beat seconds?

Ex. 339. Find the mean density of the earth, assuming the volume of the earth to be equal to that of a sphere 7913 miles in diameter. *Ans.* 5.527 times the density of water.

Ex. 340. Find the diameters of two equal spheres of lead whose attraction for each other when in contact would be 1 lb.

Ex. 341. A conical pendulum vibrates between the limits of 5° and 10°. Find the azimuth angle between these limits.

Ex. 342. A conical pendulum having $l = 30$ ft. is swinging between the limits $\alpha = 15°$ and $\beta = 6°$. How long does it take the ellipse of approximation to make one complete revolution?

Ex. 343. Assume the central ellipsoid to be given by $a=6$, $b=5$, $c=4$, and construct the herpolhode for $\delta \cos \phi = b$, from $\theta=0$ to $\theta=4\pi$. Also compute ρ for 100π.*

4. Let $b>p>c$. The discussion is similar to that for $a>p>b$, the axis of the rolling cone being the shortest axis of the ellipsoid.*

5. Let $p=c$. The discussion is similar to that for $p=a$.*

Ex. 344. (See "Notes on Rotation").

For *Ex.'s* 345 to 362, incl., see Impact, "Notes on Rotation."

The ratio of the change of momentum in the period of restitution to that in the period of compression, always less than unity, is called the coefficient of restitution. This coefficient approaches unity as the bodies approach perfect elasticity. (Denoted by e.)

Ex. 345. An ivory ball is dropped from a height of 48 in. on a fixed horizontal surface, and it rebounds to a height of 30 in. Find e.

Ex. 346. A steel ball is dropped from a height of 48 in. on a fixed horizontal surface, and it rebounds to a height of 33 in. Find e.

Ex. 347. Impact direct and central. $M_1=10$, $v_1=20$, $M_2=15$, $v_2=5$, $e=\frac{1}{3}$. Find v_3 and v_4.

Ex. 348. Impact direct and central. $M_1=2$, $v_1=30$, $M_2=20$, $v_2=8$, $e=\frac{3}{4}$. Find v_3 and v_4.

Ex. 349. Impact direct and central. $M_1=10$, $v_1=20$, $M_2=15$, $v_2=-5$, $e=\frac{3}{5}$. Find v_3 and v_4.

Ex. 350. Impact direct and central. $M_1=12$, $v_1=4$, $M_2=3$, $v_2=-16$, $v_3=-2$, $v_4=8$. Find e.

Ex. 351. Show that the loss of energy by impact is

$$\frac{\frac{1}{2}(v_1-v_2)^2(1-e^2)M_1M_2}{(M_1+M_2)}.$$

Ex. 352. Show that the numerical gain of momentum is $\frac{2(v_1-v_2)(1+e)M_1M_2}{(M_1+M_2)-2M_1v_1}$ when the bodies are moving in the same direction before and in opposite directions after the impact.

Ex. 353. Show that the numerical loss of momentum is twice the initial momentum of the reversed body when the motion is in opposite directions before and in the same direction after impact.

Ex. 354. Show that the numerical loss of momentum is $(M_1v_1+M_2v_2)(1-e)=2M_1v_1(1-e)$ when the motion is in opposite directions both before and after impact and $M_1v_1=M_2v_2$.

Ex. 355. Two smooth homogeneous spheres of the same material, with diameters 4 and 8 and velocities inversely proportional to their masses, are moving in the same plane. They collide when the direction of the motion of each center is tangent to the other sphere. Assume $e=1$. Construct the problem and use graphical values of the component velocities in making the solution for velocities after impact.

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* (See "Notes on Rotation").

Ex. 356. A particle strikes a fixed obstacle with $\phi=30°$ and $e=\frac{1}{2}$, Construct the path of the particle.

Ex. 357. A homogeneous rectangular thin plate 12 by 24 in. weighs 10 lb. It is free and at rest, and is struck normally by a 1 lb. ball moving with a velocity of 12 ft. per second, the point of impact being 4 in. from each of two adjacent edges. Assume $e=\frac{3}{4}$.

Find the velocity of the center of mass of the plate after impact, and construct the invariable, instantaneous, and spontaneous axes.

Ex. 358. A baseball bat is 89 cm. long and weighs 1 kg. Its center of mass is 35 cm. from one end, and its k^2 with respect to a central principal axis perpendicular to its length is 712. Within what limits may the bat strike the ball and keep the spontaneous axis ("Notes on Rotation"), intersecting the bat within 20 cm. of the handle end?

Ex. 359. The ballistic pendulum is an old device for finding the velocity of a projectile. It is a heavy pendulum designed to catch and retain a projectile fired into it. The swing of the pendulum after firing is registered by a marker.

Let M_e be the *equivalent mass* of the pendulum at the point impact. Then we have, e being zero,

$$v_3=v_1-\frac{M_e}{M_1+M_e}v_1, \quad \ldots\ldots \quad (1)$$

and solving for v_1,

$$v_1=\frac{M_1+M_e}{M_1}v_3. \quad \ldots\ldots \quad (2)$$

By finding the rise of the new center of mass after impact v_3 can be determined, and then v_1 from (2).

Ex. 360. A homogeneous rectangular bar 6 ft. long is mounted horizontally on a principal axis through the center of mass and perpendicular to the length of the bar, about which it can rotate without friction. The bar weighs 100 lb. It is struck by an ounce bullet at a point 1 ft. from the end, perpendicular to a vertical face, and with a velocity of $2200\frac{f}{s}$, the bullet lodging in the bar. How many revolutions per minute will the bar make after the blow?

Assume the thickness of the bar to be so small as to have a negligible effect on k.

Ex. 361. A homogeneous rectangular bar 6 ft. long weighs 80.5 lb. It is suspended from knife edges at its upper end, and is struck horizontally by an ounce bullet at a point 2 ft. from its lower end. The bullet strikes with a velocity of $2200\frac{f}{s}$, the direction of the blow being perpendicular to the line of the knife edges. The bar is penetrated and the bullet passes on with a velocity of $200\frac{f}{s}$. Through what angle will the bar oscillate after the blow?

Assume the thickness of the bar to be so small as to have a negligible effect on k.

Ex. 362. A billet 6 ft.×6 in.×6 in. is struck by a ball (solid sphere) of the same material 6 in. in diameter. The blow is direct and central with respect to the ball and normal to a face of the billet at the middle point of a 6-in. edge. The billet is free and at rest, and one second after impact the billet strikes the ball with the opposite edge of the struck face, this face then being parallel to its position when struck by the ball. Find e and the motion of the system after the second impact.

Solution. First Impact. The conditions of the problem give (logarithms in brackets),

$$M_1=1;$$

$$M_2=\frac{72}{\pi}=22.918[1.3601816];$$

$$k^2=3.02083[0.4801268];$$

$$l-h=1.00694[0.0030055];$$

$$l=4.00694[0.6028129];$$

$$\omega=\pi[0.4971509];$$

$$v_2=0.$$

$$v_4=(l-h)\pi=3.16342[0.5001564];$$

$$v_3=v_4+1=4.16342[0.6194502].$$

Then from $M_1(v_1-v_3)=M_2v_4=72.5[1.8603380]$ we have $v_1=76.66342[1.8845882]$.

For the *equivalent mass* at the point of impact,

$$M_e=\frac{M_2[k^2+(l-h)^2]}{l^2}=5.75936[0.7603745].$$

Then we have, checking the value of $(1+e)$ by solving both of the equations for direct impact,

$$1+e=\frac{(v_1-v_3)(M_e+1)}{M_ev_1}=\frac{(l\pi)(M_e+1)}{v_1}=1.1099[0.045281],$$

from which

$$e=0.1099.$$

This solution is checked by the conservation of moments as follows:

$$3M_1v_3+M_2k^2\omega=3M_1v_1=12.49026+217.5=229.99026.$$

Second impact.

$$M_1=5.75936;$$

$$M_2=1;$$

$$v_1=l\pi=12.5882[1.0999638];$$

$$v_2=4.16342.$$

The equations for direct impact give

$$v_3=11.20484,$$

$$v_1=12.13068,$$

$$\omega_2=\frac{v_3}{l}=2.79636[0.4465929].$$

The check by conservation of moments gives

$$12.13068\times3+22.918k^2\omega_2=76.66342\times3=36.39204$$
$$+193.5982=229.99024.$$

(This solution is carried to five decimal places to show the accuracy of the check.)

Ex. 363. In a *hydraulic press* the pressure is applied by a force pump having a piston 1 in. in diameter, and the piston attached directly to the platen where the pressure is utilized is 18 in. in diameter. The pump is operated by a lever which gives an advantage of 5 to 1 in the application of the power. What is the pressure at the platen for 100 lb. on the lever?

Ex. 364. A *hydraulic jack* is a form of hydraulic press used as a portable lifting machine. It is telescopic in its action, and the pump piston is located in and concentric with the large piston or ram. The ram contains a reservoir from which liquid is pumped into the pressure cylinder, and an outer pipe attached to the head of the ram surrounds the pressure cylinder and carries a lifting claw at its lower end. This claw is used to take the lift when the vertical distance available under the load is less than the height of the jack. The outer pipe protects the ram by preventing exposure of its working parts. The pump is operated by means of a detachable lever fitting in a socket attached to the side of the head.

Find the lifting pressure in the following case:

Diameter of the pump piston, $1\frac{1}{4}$ in.; diameter of the ram piston, $3\frac{1}{2}$ in.; advantage of the lever, 10 to 1; force applied to the lever, 50 lb.

Ex. 365. When a great natural head is not available the water used in working hydraulic machinery on a large scale is generally pumped into accumulators, from which it is delivered by means of valves operated at the machines. The machines are intermittent in their use, and the pumps are constantly restoring the potential of the accumulators, in the same sense that an engine restores the energy of a fly-wheel when subject to sudden reductions by heavy intermittent work. An accumulator may consist of a heavily weighted vertical ram working in a stationary cylinder, or the reverse. The weight rises while storing water in the cylinder and falls while delivering to the machines, thus furnishing a rapid supply under practically constant pressure.

What is the equivalent head of an accumulator working under a pressure of 1000 lb. per square inch?

Ex. 366. Find the pressure on the interior surface of a hollow sphere filled with water.

Find the resultant of this pressure, including its point of application.

Ex. 367. Find the pressure on the interior surface of a hollow cube filled with water, bottom horizontal.

Find the resultant of this pressure, including its point of application.

Ex. 368. A rectangular plane area is immersed with one side in the free surface of the liquid.

(*a*) Draw a horizontal line dividing the area into two parts having equal pressures.

(*b*) Draw a straight line from one of the upper corners dividing the area into two parts having equal pressures.

(*c*) Draw a straight line from one of the lower corners dividing the area into two parts having equal pressures.

Ex. 369. The *specific gravity* of a body, or its relative density, is the ratio of its weight to the weight of an equal volume of standard water (distilled water at 4° C.). The principle of the buoyant effort gives a method of obtaining the specific gravity of a body, since the indicated loss of weight when suspended in water is the weight of an equal volume of water. Except in very accurate determinations the weight in air may be taken as the weight in vacuo, since the buoyancy of air is only $\frac{2}{100}$ of a grain per cubic centimeter.

1. A Nicholson hydrometer, Fig. 55, requires a weight of 500 grains in the upper scale pan to immerse it in standard water to the fixed mark on the stem. With a specimen of brass in the upper pan it requires 300 grains to immerse it to the same depth. With the specimen in the lower pan the instrument is immersed to the mark with 324 grains in the upper pan. What is the specific gravity of the brass?

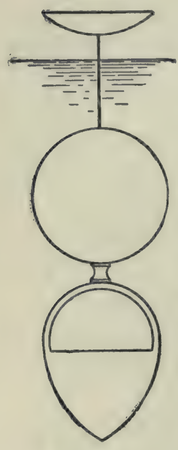

FIG. 55.

2. The instrument in (1) sinks to the mark with a piece of paraffin and $454\frac{1}{2}$ grains on the upper pan. With the paraffin secured at the lower pan it requires $504\frac{1}{2}$ grains to sink the instrument to the mark. What is the specific gravity of the paraffin?

3. A hydrometer for measuring the density of liquids weighs 800 grains and has a top scale pan. It requires 200 grains in the pan to sink to the mark in standard water. In proof spirit it sinks to the mark with 116 grains. What is the relative density of the proof spirit?

4. The instrument in (3) sinks to the mark in sea water with 225 grains in the pan. What is the relative density of the sea water?

Ex. 370. A bucket of water is placed on a scale platform and weighs 70 lb.

1. What does it weigh while a piece of lead weighing $56\frac{3}{4}$ lb. is suspended so as to be wholly immersed in the water without touching the bucket? (Density of lead is 11.35, water, unity.)

2. What does it weigh while a 6-in. cube of lead is suspended in the same manner?

3. What does it weigh while a 6-in. cube of steel is similarly suspended?

Ex. 371. Equal volumes of silver and wax, stuck together, weigh 1140 grains in vacuo and 940 grains in standard water. The density of silver is $10\frac{1}{2}$. What is the density of the wax?

Ex. 372. Equal weights of silver and wax, stuck together, weigh 1890 grains in vacuo and 750 grains in standard water. Find the density of the wax.

Ex. 373. Find the center of pressure on a parallelogram having one side in the free surface of the liquid.

Ex. 374. Show that the center of pressure on a triangle having one side in the free surface of the liquid is the middle point of the median to the lower vertex. Fig. 56, *abc*.

(This median is the locus of the centers of pressure on the linear elements parallel to the upper side, and the straight line from one of the upper vertices to a point on the opposite side at two-thirds the depth of the lower vertex is the corresponding locus for another series of parallel elements. The center of pressure on the triangle is at the intersection of these loci.)

Ex. 375. A triangle has one vertex in the free surface of the liquid and the opposite side horizontal. Find the center of pressure. Fig. 56, *bcd*.

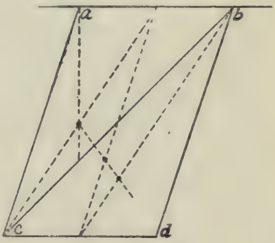

FIG. 56.

1. By the center of gravity of the superincumbent liquid.

2. By completing a parallelogram (*cabd*) of which the given triangle is one-half, finding the centers on the other half and on the parallelogram by Exs. 373 and 374, and then intersecting the line through these centers by the median from the upper vertex of the given triangle.

3. By moments, instead of by drawing the median through *b*.

Ex. 376. Explain how to find the center of pressure on any given immersed triangle. (Use the principle of Ex. 307.)

Ex. 377. Explain how to find the center of pressure on any given plane area bounded by straight lines.

Ex. 378. Find the center of pressure on any given immersed circle.

Ex. 379. Find the center of pressure on any given immersed ellipse.

Ex. 380. A dam is 20 ft. thick at the top and 60 ft. at the base. It is 96 ft. high and has one vertical and one inclined plane face. It is built of granite of 2.5 density. The water is 90 ft. deep at the base. Assume the dam to act as a monolith and to resist overturning only by the moment of its weight about the outer edge of the base, fixed against sliding.

1. Find its moment of stability with the vertical face upstream.

2. Find its moment of stability with the inclined face upstream.

3. Solve (1) and (2) assuming the water to act also on the entire base by seepage.

Ex. 381. A rolling lift dam, cylindrical and of circular cross-section, rests on the sill and its upper element is at the water level. Fig. 57. (See Scientific American, vol. 90, p. 312.)

1. Find the total hydrostatic pressure on a unit length of the cylinder. What is the graphical check of this solution?

2. Find the center of pressure and give its depth.

3. Find the horizontal component of the pressure by integration.

Solution of 3. The horizontal component of the pressure on dS is $\delta gzdS \sin\theta = \delta gr^2(\sin\theta - \sin\theta\cos\theta)d\theta$, since $z = r - r\cos\theta$ and $dS = rd\theta$. Then we have for the horizontal pressure,

$$r^2\int_0^\pi \left(\sin\theta - \sin\theta\cos\theta\right)d\theta = \delta gr^2\left(-\cos\theta - \tfrac{1}{2}\sin^2\theta\right)_0^\pi = 2\delta gr^2.$$

For the moment of $\delta gzrd\theta \sin\theta$ with respect to the upper element of the cylinder we have $\delta gz^2rd\theta\sin\theta = \delta gr^3(\sin\theta - 2\sin\theta\cos\theta +$

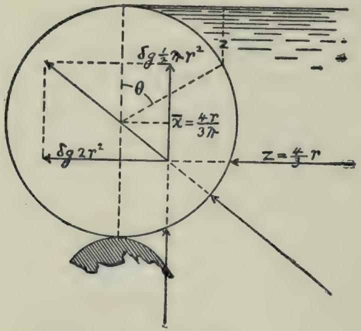

FIG. 57.

$\sin\theta\cos^2\theta)d\theta$, and the required integral for the resultant moment of the horizontal pressure is

$$\delta gr^3\left(-\cos\theta - \sin^2\theta - \tfrac{1}{3}\cos^3\theta\right)_0^\pi = \tfrac{8}{3}\delta gr^3.$$

Then we have for the lever arm of the horizontal pressure,

$$\frac{\frac{8}{3}\delta gr^3}{2\delta gr^2} = \tfrac{4}{3}r,$$

thus giving the depth of the center of horizontal pressure.

Ex. 382. The displacement of a ship to the Plimsoll mark in fresh water is 10,000 tons, and the water-line area is 25,000 sq. ft. How much does the ship rise in going into still sea water? (Assume the sides of the ship to be vertical at the water-line.)

Ex. 383. An isosceles wedge floats with back horizontal and edge down. Show that the width of the water-line area is equal to the width of the back multiplied by the square root of the mean density of the wedge. (The same factor applies to other similar dimension of the wedge and its immersed part.)

Ex. 384. A right cone with a circular base floats with axis vertical and vertex down. Show that the radius of the water-line area is equal to the radius of the base multiplied by the cube root of the mean density of the cone. (The same factor applies to other similar dimensions.)

Ex. 385. Find the condition for stability of a homogeneous rectangular parallelopiped floating with a face horizontal; length a, width b, depth c.

Solution for Axis Parallel to a.

$$V = \delta abc;\quad A = ab;\quad Ak^2 = \tfrac{1}{12}ab^3;$$

$$\frac{Ak^2}{V} = \frac{b^2}{12\delta c};\quad BG = \tfrac{1}{2}c - \tfrac{1}{2}\delta c = \tfrac{1}{2}c(1-\delta);$$

$$\tfrac{1}{2}c(1-\delta) < \frac{b^2}{12\delta c},\ \text{or}\ b > c\sqrt{6\delta(1-\delta)}.$$

Ex. 386. Show that the condition for stability of a homogeneous right circular cylinder floating vertically, radius r and length l, is $r > l\sqrt{2\delta(1-\delta)}$.

Ex. 387. The limit of stability for a homogeneous right circular cylinder floating vertically is $r > \frac{2}{3}l$. Find the density of the cylinder.

Ex. 388. A homogeneous isosceles wedge floats with back horizontal and edge down; length a (parallel to edge), width b, depth c.

(1) Show that the condition for stability is $b > 2c\sqrt{\frac{(\delta^{1/2} - \delta)}{\delta}}$ when the plane of θ is normal to a.

(2) Show that the condition for stability is $a > 2c\sqrt{\delta^{1/2} - \delta}$ when the plane of θ is normal to b.

3. Show from the conditions given in 1 and 2 that when the wedge is equally stable in the two cases the water-line area is square.

4. Show from the measure of stability, $Ak^2 - V\ (BG)$,

A = water-line area;
K^2 = radius of gyration for A;
V = volume of displacement;
BG = distance from point of application of buoyant effort B to the the center of gravity G,

that the wedge is equally stable in the two cases when the water-line area is square.

5. Construct the metacenter for case (1). Fig. 58.

The value of $Bm=\frac{Ak^2}{V}=\frac{\frac{1}{6}b^2\delta^{1/2}}{c}$ (m indicates metacenter) may be factored to read $(\frac{1}{3}b\delta^{1/2})\left(\frac{\frac{1}{2}b}{c}\right)$. The first factor in parenthesis is one-third of the width of the water-line area, or one-half the width of the horizontal section through the center of buoyancy; and the second is the tangent of the semiangle of the wedge. Therefore, draw through the center of buoyancy a parallel to b, and at the point where it meets the face of the wedge draw a normal. This normal cuts the vertical through the center of gravity at the metacenter, the angle between the horizontal and normal being equal to the semiangle of the wedge.

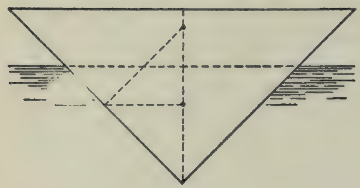

FIG. 58.

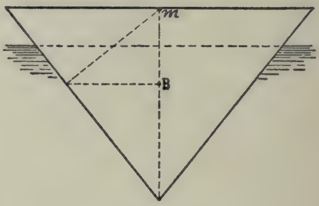

FIG. 59.

6. Find the value of δ for the given figure.

7. Find the condition for stability of the wedge inverted, θ normal to a.

8. Find the condition for stability of the wedge inverted, θ normal to b.

Ex. 389. A homogeneous right circular cone floats with its base horizontal and vertex down; radius of base r, axial height h.

1. Show that the condition for stability is $r>h(\delta^{-1/3}-1)^{1/2}$.

(2) Show that the position of the metacenter is given by the construction in Fig. 59, similar to that of Fig. 58, factoring the value of $\frac{Ak^2}{V}$ to read $\left[(\frac{3}{4}h\delta^{1/3})\left(\frac{r}{h}\right)\right]\left(\frac{r}{h}\right)$.

3. Find the value of δ for Fig. 59.

4. Show that the condition for stability is $r>h[(1-\delta)^{-1/3}-1]^{1/2}$ when the cone floats vertically with the vertex up.

Ex. 390. Show that if a homogeneous body be stable on any water-line, it will be stable on the same water-line if inverted, provided its density be changed from δ to $1-\delta$.

Let δ_1 and V_1 be the density and the volume of the displacement when erect, and δ_2 and V_2 the corresponding density and volume when inverted. Then, since V_1+V_2 is the volume of the body, we have for the displacement volumes,

$$V_1=\delta(V_1+V_2),$$

$$V_2=\delta_2(V_1+V_2);$$

and by addition and reduction we get

$$\delta_2=1-\delta_1.$$

Ex. 391. Show that two homogeneous bodies of the same size and form are equally stable when they float inverted with respect to each other and their densities are δ and $1-\delta$. (See Ex. 390.)

For the measure of stability in the two cases we have,

$$A_1k_1^2-V_1(B_1G), \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$A_2k_2^2-V^2(B^2G). \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

The water-line areas are identical in form and size, making $A_1k_1^2=A_2k_2^2$; and, from the principle of the center of gravity, $V_1(B_1G)=V_2(B_2G)$. Therefore the two measures are equal.

Ex. 392. Show that the metacentric heights are inversely as the densities when two homogeneous bodies of the same size and form are stable on the same water-line with densities δ and $1-\delta$.

a. By Ex. 391 the ratio of their measures is unity. Multiply this ratio by $\frac{V_2}{V_1}=\frac{\delta_2}{\delta_1}$ (Ex. 390), giving

$$\frac{\left(\frac{Ak^2}{V_1-B_1G}\right)}{\left(\frac{Ak^2}{V_2-B_2G}\right)}=\frac{\delta_2}{\delta_1}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

(*b*) Show this by analysis, using the fact of equal stability and the relation of the metacentric height to the lever arm of the buoyant effort.

Ex. 393. Assume the water main in Fig. 60 to be 6 ft. in diameter and the mercury in the gauge to stand at 7 in. Find the discharge per minute, in gallons. (Density of mercury = 13,596; 1 gal. = 231 cu. in.)

Ex. 394. Apply Bernouilli's Law to the pipes in Fig. 61, so as to check the velocity head for each.

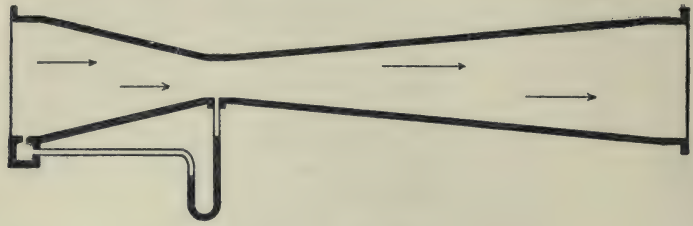

FIG. 60.

Ex. 395. **The Form of the Free Surface of a Rotating Fluid.** Take the axis of rotation to be Z and the angular velocity constant.

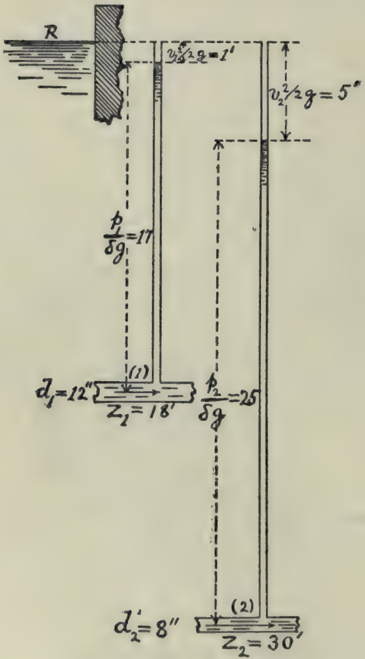

FIG. 61.

Then the resultant acceleration of any particle is $\frac{v^2}{\rho}$, and its components are

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=\left(\frac{v^2}{\rho}\right)\cos\alpha=\left(\frac{\omega^2\rho^2}{\rho}\right)\left(\frac{-x}{\rho}\right)=-\omega^2x,\\ \frac{d^2y}{dt^2}&=\left(\frac{v^2}{\rho}\right)\cos\beta=\left(\frac{\omega^2\rho^2}{\rho}\right)\left(\frac{-y}{\rho}\right)=-\omega^2y,\\ \frac{d^2z}{dt^2}&=\left(\frac{v^2}{\rho}\right)\cos\gamma=0.\end{aligned}\right\}\quad(1)$$

These in the differential equations of flow, give

$$\left.\begin{aligned}\frac{\left(\dfrac{\partial P}{\partial x}\right)}{\delta}&=g\cos\alpha+\omega^2x,\\ \frac{\left(\dfrac{\partial P}{\partial y}\right)}{\delta}&=g\cos\beta+\omega^2y,\\ \frac{\left(\dfrac{\partial P}{\partial z}\right)}{\delta}&=g\cos\gamma.\end{aligned}\right\}\quad\ldots\ldots\quad(2)$$

Multiplying each by its coordinate differential and adding we get

$$\frac{dP}{\delta}=(g\cos\alpha+\omega^2x)dx+(g\cos\beta+\omega^2y)dy+g\cos\gamma dz.\quad .\quad(3)$$

For the free surface $dP=0$, and its differential equation is therefore

$$(g\cos\alpha+\omega^2x)dx+(g\cos\beta+\omega^2y)dy+g\cos\gamma dz=0.\quad .\quad(4)$$

This equation is also applicable to any surface of constant pressure.

Ex. 395 (*a*). Find the free surface of a liquid in a vertical cylinder rotating about its axis with constant angular velocity. (Fig. 62.)

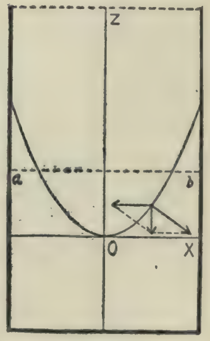

FIG. 62.

Solution. The differential equation of the surface Eq. (4), Ex. 395, becomes

$$gdz=\omega^2xdx+\omega^2ydy,\quad\ldots\ldots\quad(1)$$

since $g\cos\alpha=g\cos\beta=0$ and $g\cos\gamma=-g$.

By integration

$$gz=\tfrac{1}{2}\omega^2(x^2+y^2)+C.\quad\ldots\ldots\quad(2)$$

Taking the origin at the point where the axis of z pierces the free surface we have $C=0$, and (2) becomes

$$\omega^2(x^2+y^2)=2gz, \qquad (3)$$

which is the equation of a paraboloid of revolution. Its section in the XZ plane is shown in the figure.

The relation of this surface to the free surface at rest may be shown, as follows:

For the xz section of the free surface in motion we have

$$x^2=\frac{2gz}{\omega^2} \qquad (4)$$

and the volume of liquid above the xy plane (r being the radius of the cylinder and z_r the extreme height of the free surface in rotation) is given by

$$\int_0^r 2\pi x dx \cdot z=\frac{\pi\omega^2}{g}\int_0^r x^3 dx=\frac{\frac{1}{4}r^4\pi\omega^2}{g}=\tfrac{1}{2}\pi r^2 z_r. \qquad (5)$$

That is, the volume of liquid above the xy plane is half the volume of the cylinder between z_0 and z_r. Therefore the free surface at rest is at $\frac{1}{2}z_r$, *ab* in the figure.

From Eq. (4) we see that the velocity of any particle of the free surface is the velocity due to its height above the lowest point of the surface.

In the force diagram in the figure (for a particle at the free surface) the horizontal component of the weight of the particle is the force that produces the resultant acceleration, $\frac{v^2}{\rho}$; the other component of the weight being equal and directly opposed to the normal recation of the surface (liquid) against the particle.

NOTES ON TRANSLATION AND ROTATION

These notes deal with the mathematical development of the subject from the energy basis.

They are from "Gordon's Mechanics." Colonel Gordon's treatment of the subject is more complete than the discussions found in most texts for use by students. It is, therefore, included as a reference, although not forming a part of the present course.

1. **The Law of Energy for a Conservative System.** The general law of energy for such a system is formulated as follows:

The masses of the system are made up of particles, each of which is subject to the law of change which has just been explained. Limiting the statement of the law to the elementary changes of energy, we have for each particle and the force transmitted to it,

$$Fdp = m\left(\frac{d^2s}{dt^2}\right)ds, \qquad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

and for the entire system,

$$\Sigma Fdp = \Sigma m\left(\frac{d^2s}{dt^2}\right) ds, \qquad . \; . \; . \; . \; . \; . \; . \quad (E)$$

which is the general equation of energy for any conservative system. It is also called the *law of conservation of energy,* since for any loss or gain of potential there is an equal gain or loss of kinetic energy.

The simplicity of this general law arises from the fact that work and energy are scalar quantities, and, therefore, the work done on an entire system can be summed up arithmetically, as in the equation without regard to direction.

The law was deduced by considering the forces applied to each particle separately, but this fact imposes no limitation on the first member of the equation. The conditions would be actually as assumed in a system of unconnected free particles, but in the case of a rigid·solid the forces may be applied at particular points, or to each particle separately, or both of these conditions may exist simultaneously. A force applied at a particular point would transmit a certain intensity to each particle of the body, and the work done by the single applied force would necessarily be equal to the aggregate of the work done by its transmitted components. We therefore take the first member of Eq. (E) to be the work of the applied forces, however given; and the second member to be the corresponding change in the kinetic energy of the system, however it may be distributed among the particles.

For example, in the case of the trajectory in air, the resistance of the air is applied directly to the surface of the projectile; but a certain stress is transmitted to each particle, and the aggregate work of these transmitted stresses is equal to the work of the resultant resistance over its own path. At the same time the attraction of the earth is applied to each particle directly, and the aggregate work of these molecular weights is equal to the work of the resultant weight over the vertical path of the center of gravity. Thus, between any limits, $\Sigma\int_1^2 Fdp$ includes the work of the weight of the projectile and the resistance of the air, while the change in the kinetic energy between the same limits is

$$\Sigma\int_2^1 m\left(\frac{d^2s}{dt^2}\right)ds.$$

There are as many terms in the first member of Eq. (E) *as there are forces in the system, and as many in the second member as there are particles in the bodies of the system.* The equation must, therefore, be simplified before we can make its application to the general case of the motion of a rigid solid.

Before making the necessary transformation for this purpose we will examine more closely into the character of forces in general than was necessary in the subject of statics. In the present state of scientific knowledge we cannot analyze the ultimate relation of force to matter, nor even decide the question as to their separate identity; and neither force nor matter can be defined with scientific accuracy. We simply use the term " matter " to refer to the substance of *any material body*, without regard to *kind;* and when we wish to speak of matter in a quantitative sense we use the term " mass."

Likewise, when we refer to the invisible agency which accelerates matter—or tends to accelerate it—we use the term " force "; and force is expressed quantitatively in terms of a unit which gives unit acceleration to a unit of mass.

In statics the tendency to accelerate is counteracted by the equal and opposite tendency of other forces, and the effects are strains instead of accelerations.

In the free motions of a conservative system, however, force has an additional significance—or interpretation—as follows:

Force Defined as a Rate; Momentum. For the resultant force applied to a free particle we may write

$$F=\frac{(Fdp)}{dp}=m\left(\frac{dv}{dt}\right)=\frac{d(mv)}{dt}. \qquad . \quad . \quad . \quad . \quad . \quad (2)$$

The quantity *mv*, which is the product of a mass by its velocity, is called *momentum;* and from Eq. 2 we see that a force may be defined as either *the rate of change of the potential as a function of the path*, or *the rate of change of the momentum as a function of the time.*

It must be remembered that both of these rates are in the direction of *F*, and that *mv* is a vector quantity.

There is no inconsistency in the various ways of estimating a force; *as a simple pressure, in the product of a free mass by its acceleration, by the space rate of change of the energy, and by the time rate of change of the momentum.* They are all convertible into the same numerical value, which, in its final conception, is the measure of a stress in some medium, in effect like a tension or compression. Thus, a ball whirled at the end of a string has a force applied through the tension of the string; a ball struck by a bat, by the compression of the bat; an attraction, such as that between the earth and the moon, by a stress (similar to tension) in the medium which contains and permeates both bodies, and a magnetic or electric repulsion by an opposite stress in the same medium. The static or dynamic effect in a particular case may be said to be a measure of the given force, while the force itself is presented to the mind as a stress due to a condition of strain in a medium. It is not known how this condition is produced in electric, magnetic, and gravitational forces, but as far as the mechanical effect is concerned it is not necessary to go beyond the conception of the force and its relation to the mass and acceleration of the body on which it acts.

Strictly, it seems to be illogical to speak of any part of the phenomenon as *cause* and another part as *effect,* since we can make no analysis beyond the observed relations between forces, masses, and accelerations. But following the apparent analogy of a force visibly applied, as in the motion imparted to a ball by a bat, there need be no confusion of ideas in speaking of mass as inert, and of force as the cause of acceleration. "The correct view—the one which justifies the idea of force—is that which regards force as if it were a persistent cause, and always producing the same effect under the same circumstances."*

* Helmholtz.

2. Reference of Eq. (E) to Rectangular Coordinate Axes. The first step in the simplification of the equation of energy is to refer it to rectangular coordinate axes. Since it makes no difference whether we consider work to be done directly by a force, or by the components of the force acting separately, we may write

$$FdP=F_x dx+F_y dy+F_z dz. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

So also, for the work of the resultant of the forces applied to each particle of a system, either directly or by transmission we have

$$m\left(\frac{d^2s}{dt^2}\right)ds=m\left(\frac{d^2x}{dt^2}\right)dx+m\left(\frac{d^2y}{dt^2}\right)dy+m\left(\frac{d^2z}{dt^2}\right)dz, \quad . \quad (4)$$

since the $\frac{mv^2}{\rho}$ component does no work. We may therefore write, instead of $\Sigma Fdp=\Sigma m\left(\frac{d^2s}{dt^2}\right)ds$,

$$\Sigma F\cos\alpha dx+\Sigma F\cos\beta dy+\Sigma F\cos\gamma dz= \Sigma m\left(\frac{d^2x}{dt^2}\right)dx+\Sigma m\left(\frac{d^2y}{dt^2}\right)dy+\Sigma m\left(\frac{d^2z}{dt^2}\right)dz, \quad (5)$$

for the equation of energy referred to rectangular coordinate axes.

Eq. (5) *deduced analytically.* Eq. (3) may also be obtained by writing

$$dp=dx\cos\alpha+dy\cos\beta+dz\cos\gamma, \quad . \quad . \quad . \quad . \quad (6)$$

and multiplying by F; and (4) by differentiating

$$\frac{ds^2}{dt^2}=\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and multiplying by $\frac{m}{2}$. Eq. (5) therefore follows analytically, without direct reference to the principle of work.

3. Application of the Equation of Energy to the Motion of a Rigid Solid. The assumption of rigidity in a body to which forces are applied is merely for the purpose of avoiding consideration of the distortion produced by the forces, and thus to simplify the problem of motion by neglecting the displacements of the particles relative to each other. We then assume the most general motion of the body consistent with this limitation, and substitute in Eq. (5) the corresponding values of the differentials of the variables.

Any motion of a rigid solid may be divided into two parts, namely: the linear motion of any particular point of the body, and the rotation of the body about that point.

Thus, in Fig. 1 let m be any particle of a rigid body, and let O' be a particular point of the body, selected to define the *motion of translation*. Then, *under the action of any forces whatever*, the motion of the body consists of the linear motion of O' and the rotation of the body about O'.

The axes X, Y, Z, are fixed in position, and O is called *the fixed origin*. The axes X', Y', Z', pass through O' and are parallel to the X, Y, Z axes, and O' is called *the movable origin*. Then, according to the notation indicated in the figure, the coordinates of m referred to the fixed origin are

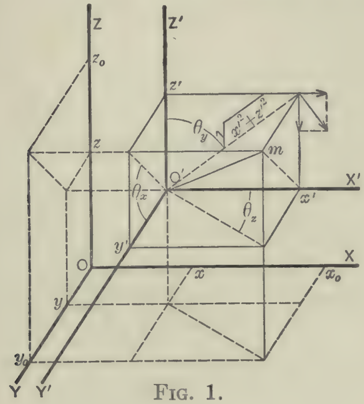

FIG. 1.

$$\left.\begin{aligned} x_0&=x+x',\\ y_0&=y+y',\\ z_0&=z+z', \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

and their differentials are

$$\left.\begin{aligned} dx_0&=dx+dx',\\ dy_0&=dy+dy',\\ dz_0&=dz+dz'. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

The coordinates x', y', z', and their differentials, depend on the rotation of the body about O', and this rotation is defined by the component rotations about X', Y', Z', as indicated by the variable angles θ_x, θ_y, θ_z. An increment of each angle gives corresponding increments to x', y', z', and the partial differential of these coordinates are functions of $d\theta_x$, $d\theta_y$, $d\theta_z$. Thus, in the figure, as shown on the XZ plane, we have

$$\partial x' = \sqrt{x'^2+z'^2}d\theta_y \cos \theta_y = z'd\theta_y,$$
$$\partial z' = \sqrt{x'^2+z'^2}d\theta_y \sin \theta_y = -x'd\theta_y,$$

while for rotation about Y' there is no change in y', making $\partial y' = 0$.

Increments of θ_x and θ_z give similar values, and we thus have the following groups, arranged in order:

For a positive increment of θ_x,

$$\left.\begin{aligned}\partial x' &= 0,\\ \partial y' &= \sqrt{y'^2+z'^2}d\theta_x \sin \theta_x = -z'd\theta_x,\\ \partial z' &= \sqrt{y'^2+z'^2}d\theta_x \cos \theta_x = y'd\theta_x.\end{aligned}\right\} \quad . \quad . \quad . \quad (10)$$

For a positive increment of θ_y,

$$\left.\begin{aligned}\partial x' &= \sqrt{x'^2+z'^2}d\theta_y \cos \theta_y = z'd\theta_y.\\ \partial y' &= 0,\\ \partial z' &= \sqrt{x'^2+z'^2}d\theta_y \sin \theta_y = -x'd\theta_y.\end{aligned}\right\} \quad . \quad . \quad . \quad (11)$$

For a positive increment of θ_z,

$$\left.\begin{aligned}\partial x' &= \sqrt{x'^2+y'^2}d\theta_z \sin \theta_z = -y'd\theta_z,\\ \partial y' &= \sqrt{x'^2+y'^2}d\theta_z \cos \theta_z = x'd\theta_z,\\ \partial z' &= 0.\end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

Therefore we have for the total differentials,

$$\left.\begin{aligned}dx' &= z'd\theta_y - y'd\theta_z,\\ dy' &= x'd\theta_z - z'd\theta_x,\\ dz' &= y'd\theta_x - x'd\theta_y,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and substituting these values in (9) we have

$$\left.\begin{aligned}dx_0 &= dx + z'd\theta_y - y'd\theta_z,\\ dy_0 &= dy + x'd\theta_z - z'd\theta_x,\\ dz_0 &= dz + y'd\theta_x - x'd\theta_y.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

These values are general for any particle of the body, and we may substitute them in Eq. (5) for dx, dy, dz. Then, changing the acceleration factors to correspond with this change in the notation we have (x, y, z, becoming x_0, y_0, z_0),

$$\left.\begin{array}{l} \Sigma F_x dx + \\ \Sigma F_y dy + \\ \Sigma F_z dz + \\ \Sigma(F_z y' - F_x z')d\theta_x + \\ \Sigma(F_x z' - F_z x')d\theta_y + \\ \Sigma(F_y x' - F_x y')d\theta_z \end{array}\right\} = \left\{\begin{array}{l} \Sigma m\left(\dfrac{d^2x_0}{dt^2}\right)dx + \\ \Sigma m\left(\dfrac{d^2y_0}{dt^2}\right)dy + \\ \Sigma m\left(\dfrac{d^2z_0}{dt^2}\right)dz + \\ \Sigma\left[m\left(\dfrac{d^2z_0}{dt^2}\right)y' - m\left(\dfrac{d^2y_0}{dt^2}\right)z'\right]d\theta_x + \\ \Sigma\left[m\left(\dfrac{d^2x_0}{dt^2}\right)z' - m\left(\dfrac{d^2z_0}{dt^2}\right)x'\right]d\theta_y + \\ \Sigma\left[m\left(\dfrac{d^2y_0}{dt^2}\right)x' - m\left(\dfrac{d^2x_0}{dt^2}\right)y'\right]d\theta_z. \end{array}\right\} \quad (15)$$

Dividing by dt and changing the notation for the component linear velocities of the movable origin, angular velocities about the movable axes and component moments of the forces, we have

$$\left.\begin{array}{l} \Sigma F_x v_x + \\ \Sigma F_y v_y + \\ \Sigma F_z v_z + \\ \Sigma M_{x'} \omega_x + \\ \Sigma M_{y'} \omega_y + \\ \Sigma M_{z'} \omega_z \end{array}\right\} = \left\{\begin{array}{l} \Sigma m\left(\dfrac{d^2x_0}{dt^2}\right)v_x + \\ \Sigma m\left(\dfrac{d^2y_0}{dt^2}\right)v_y + \\ \Sigma m\left(\dfrac{d^2z_0}{dt^2}\right)v_z + \\ \Sigma\left[m\left(\dfrac{d^2z_0}{dt^2}\right)y' - m\left(\dfrac{d^2y_0}{dt^2}\right)z'\right]\omega_x + \\ \Sigma\left[m\left(\dfrac{d^2x_0}{dt^2}\right)z' - m\left(\dfrac{d^2z_0}{dt^2}\right)x'\right]\omega_y + \\ \Sigma\left[m\left(\dfrac{d^2y_0}{dt^2}\right)x' - m\left(\dfrac{d^2x_0}{dt^2}\right)y'\right]\omega_z. \end{array}\right\} \quad (16)$$

Having any system of forces acting on any rigid body, we may assume the initial conditions of motion to be given by arbitrarily assigned values of the component linear and angular velocity factors in this equation. That is, the equation must be true whatever be the values of v_x, v_y, v_z, ω_x ω_y, ω_z. We, therefore, have, by the principle of undetermined coefficients,

$$\left.\begin{aligned}\Sigma F_x &= \Sigma m\left(\frac{d^2x_0}{dt^2}\right),\\ \Sigma F_y &= \Sigma m\left(\frac{d^2y_0}{d^2t}\right),\\ \Sigma F_z &= \Sigma m\left(\frac{d^2z_0}{dt^2}\right),\end{aligned}\right\} \quad \dots\dots\dots \quad (T)$$

$$\left.\begin{aligned}\Sigma M_{x'} &= \Sigma\left[m\left(\frac{d^2z_0}{dt^2}\right)y' - m\left(\frac{d^2y_0}{dt^2}\right)z'\right],\\ \Sigma M_{y'} &= \Sigma\left[m\left(\frac{d^2x_0}{dt^2}\right)z' - m\left(\frac{d^2z_0}{dt^2}\right)x'\right],\\ \Sigma M_{z'} &= \Sigma\left[m\left(\frac{d^2y_0}{dt^2}\right)x' - m\left(\frac{d^2x_0}{dt^2}\right)y'\right].\end{aligned}\right\} \quad \dots \quad (R)$$

The first three of these equations relate to the motion of translation of the movable origin, and the second set of three to the rotation of the body about the movable axes. Hence, their designation as equations (T) and (R).

4. Reference of Eqs. (T) and (R) to the Center of Mass as a Movable Origin. Eqs. (T) and (R) are not of such form as to be capable of direct application, since each of the second members is the summation of products whose separate values cannot be found directly; but by using the center of mass as a movable origin the equations are simplified as follows:

Translation. From $x_0 = x + x'$ we have $\dfrac{d^2x_0}{dt^2} = \dfrac{d^2x}{dt^2} + \dfrac{d^2x'}{dt^2}$, and writing R_x for ΣF_x we have for the first of Eqs. (T),

$$R_x = \frac{\Sigma m d^2x}{dt^2} + \frac{\Sigma m d^2x'}{dt^2}. \quad \dots\dots \quad (17)$$

For the center of mass as an origin we have $\Sigma m x' = 0$, and hence $\dfrac{\Sigma m d^2x'}{dt^2} = 0$. This reduces (17) to

$$R_x = \frac{\Sigma m d^2x}{dt^2}, \quad \dots\dots \quad (18)$$

$\dfrac{d^2x}{dt^2}$ being the x component of the acceleration of the center of mass, and therefore a common factor in each of the products of the type $\frac{md^2x}{dt^2}$. We may therefore write $\frac{\Sigma md^2x}{dt^2}=\frac{Md^2x}{dt^2}$, M being the entire mass of the body. Eqs. (T) thus become (with similar reductions for the other axes),

$$\left.\begin{aligned} R_x&=\frac{Md^2x}{dt^2},\\ R_y&=\frac{Md^2y}{dt^2},\\ R_z&=\frac{Md^2z}{dt^2}, \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (T_{cm})$$

and solving for the component accelerations,

$$\left.\begin{aligned} \frac{d^2x}{dt^2}&=\frac{R_x}{M},\\ \frac{d^2y}{dt^2}&=\frac{R_y}{M},\\ \frac{d^2z}{dt^2}&=\frac{R_z}{M}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (T_{cm})$$

These are the equations which we have already used for the motion of a heavy particle of mass M under the action of any system of forces. We therefore conclude that, when any system of forces acts on a rigid solid, *the center of mass moves as if it were a heavy particle having a mass equal to that of the body, under the action of forces equal and parallel to the given system.* That is, the form of the body has no effect on the motion of the center of mass, and the solution of the problem of motion of the center of mass of a rigid solid is exactly the same as that of the motion of a heavy particle.

Rotation. From the first of Eq. (R) we have, by substituting $\frac{d^2x}{dt^2}+\frac{d^2x'}{dt^2}$ for $\frac{d^2x_0}{dt^2}$ and writing $M_{x'}$ for the x' component moment of the system of forces,

$$\left.\begin{aligned} M_{x'}=(\Sigma my')\left(\frac{d^2z}{dt^2}\right)+\Sigma m\left(\frac{d^2z'}{dt^2}\right)y'\\ -(\Sigma mz')\left(\frac{d^2y}{dt^2}\right)-\Sigma m\left(\frac{d^2y'}{dt^2}\right)z'. \end{aligned}\right\} \quad . \; . \; . \; . \quad (19)$$

Then taking the center of mass as the movable origin and applying the principle of the center of mass, we have for (R_{cm}), including similar reductions of M_y and M_z,

$$\left.\begin{aligned} M_{x'} &= \Sigma\left[m\left(\frac{d^2z'}{dt^2}\right)y' - m\left(\frac{d^2y'}{dt^2}\right)z'\right], \\ M_{y'} &= \Sigma\left[m\left(\frac{d^2x'}{dt^2}\right)z' - m\left(\frac{d^2z'}{dt^2}\right)x'\right], \\ M_{z'} &= \Sigma\left[m\left(\frac{d^2y'}{dt^2}\right)x' - m\left(\frac{d^2x'}{dt^2}\right)y'\right]. \end{aligned}\right\} \quad . \quad . \qquad (R_{cm})$$

The coordinates of the center of mass do not enter these equations, and, therefore, we conclude that *the motion of the center of mass and the rotation of the body about the center of mass are independent of each other, and may be treated as separate problems.*

Eqs. (R_{cm}) still contain the coordinates of the separate particles of the body, and require further simplification before they can be directly applied. This part of the problem is treated in the notes on rotation.

Conclusion. The solution of the problem of motion of a rigid solid consists of two parts, namely:

1. *Finding the position of the center of mass by Eqs.* (T_{cm}).
2. *Finding the position of the body with respect to the center of mass by Eqs.* (R_{cm}).

5. The Work of Translation and Rotation. Having obtained Eqs. (T_{cm}) and $R_{cm})$ from the equation of energy by the principle of undetermined coefficients, it is obvious that the proper path factor may be restored to each coefficient, thus giving an energy equation for each of Eqs. (T_{cm}) and $R_{cm})$.

Translation. We thus have for translation,

$$\left.\begin{aligned} R_x dx &= M\left(\frac{d^2x}{dt^2}\right)dx, \\ R_y dy &= M\left(\frac{d^2y}{dt^2}\right)dy, \\ R_z dz &= M\left(\frac{d^2z}{dt^2}\right)dz. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

For the total work of translation between limiting positions of the center of mass we have

$$\int_1^2 (R_x dx + R_y dy + R_z dz) = \tfrac{1}{2}Mv_2^2 - \tfrac{1}{2}Mv_1^2, \quad . \quad . \quad . \qquad (21)$$

v_1 and v_2 being limiting velocities of the center of mass. The work of translation is therefore the same as if the body were a particle of mass M, with the forces applied directly to it without change of intensity or direction.

Rotation. The energy equations for rotation about the center of mass are

$$\left.\begin{aligned} \Sigma M_{x'}d\theta_x &= \Sigma\left[m\left(\frac{d^2z'}{dt^2}\right)y' - m\left(\frac{d^2y'}{dt^2}\right)z'\right]d\theta_x, \\ \Sigma M_{y'}d\theta_y &= \Sigma\left[m\left(\frac{d^2x'}{dt^2}\right)z' - m\left(\frac{d^2z'}{dt^2}\right)x'\right]d\theta_y, \\ \Sigma M_z d\theta_z &= \Sigma\left[m\left(\frac{d^2y'}{dt^2}\right)x' - m\left(\frac{d^2x'}{dt^2}\right)y'\right]d\theta_z. \end{aligned}\right\} \quad . \quad . \quad (22)$$

To show that the work of the component moments is equal to the kinetic energy of rotation, add Eqs. (22), collect the coefficients of the acceleration factors, and reduce by Eqs. (13). We then have

$$\begin{aligned} &\Sigma\int_1^2 M_{x'}d\theta_x + M_{y'}d\theta_y + M_{z'}d\theta_z) = \\ &\Sigma\int_1^2\left[m\left(\frac{d^2x'}{dt^2}\right)dx' + m\left(\frac{d^2y'}{dt^2}\right)dy' + m\left(\frac{d^2z'}{dt^2}\right)dz'\right] = \\ &\tfrac{1}{2}\Sigma m v_2^2 - \tfrac{1}{2}\Sigma m v_1^2, \qquad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (23) \end{aligned}$$

v_1 and v_2 being the limiting velocities of m with respect to the center of mass.

The work of translation and of rotation are thus seen to be separate problems. The work of translation is the aggregate work of the component forces over the component paths of the center of mass, while the work of rotation is that of the component moments over the component angular paths about the movable axes through the center of mass.

The first member of (23) may be put in the form $\Sigma\int_1^2 (F_x dx' + F_y dy' + F_z dz')$ by writing the component moments in terms of the component forces and the coordinates of their points of application, as in Eq. (15), and then collecting the coefficients of the component forces and reducing by (13). This shows that the work of rotation is also the work of the given forces over their linear paths with respect to the center of mass.

Motion of Rotation

6. The First Reduction of (R_{cm}). Before applying equations (R_{cm}) to the rotation of a rigid body it is necessary to simplify the terms involving the masses of the separate particles. For this purpose assume (R_{cm}), omitting accents,

$$\left.\begin{aligned} M_x&=\Sigma m\left(\frac{yd^2z}{dt^2}\right)-\left(\frac{zd^2y}{dt^2}\right),\\ M_y&=\Sigma m\left(\frac{zd^2x}{dt^2}\right)-\left(\frac{xd^2z}{dt^2}\right),\\ M_z&=\Sigma m\left(\frac{xd^2y}{dt^2}\right)-\left(\frac{yd^2x}{dt^2}\right),\end{aligned}\right\}\quad .\ .\ .\ .\ . \quad (R_{cm})$$

and change their form by replacing the component accelerations with values obtained from Eq. (13). The object of this transformation is to introduce angular instead of linear accelerations, and to reduce the mass-coordinate factors to forms capable of separate evaluation.

The first reduction is as follows:

Assume Eq. (13), without accents,

$$\left.\begin{aligned} dx&=zd\theta_y-yd\theta_z,\\ dy&=xd\theta_z-zd\theta_x,\\ dz&=yd\theta_x-xd\theta_y,\end{aligned}\right\}\quad .\ .\ .\ .\ .\ .\ .\ . \quad (13)$$

and obtain from them the following values for the component accelerations in (R_{cm}):

$$\left.\begin{aligned} \frac{d^2x}{dt^2}&=\frac{zd\omega_y}{dt}-\frac{yd\omega_z}{dt}+\omega_y(y\omega_x-x\omega_y)-\omega_z(x\omega_z-z\omega_x),\\ \frac{d^2y}{dt^2}&=\frac{xd\omega_z}{dt}-\frac{zd\omega_x}{dt}+\omega_z(z\omega_y-y\omega_z)-\omega_x(y\omega_x-x\omega_y),\\ \frac{d^2z}{dt^2}&=\frac{yd\omega_x}{dt}-\frac{xd\omega_y}{dt}+\omega_x(x\omega_z-z\omega_x)-\omega_y(z\omega_y-y\omega_z).\end{aligned}\right\}\quad . \quad (24)$$

These values in (R_{cm}) give

$$\left.\begin{aligned}
M_x &= \Sigma m(y^2+z^2)\frac{d\omega_x}{dt} + \Sigma m(y^2-z^2)\omega_y\omega_z \\
&\quad -\Sigma myz(\omega_y{}^2-\omega_y{}^2) - \Sigma mzx\left(\frac{d\omega_z}{dt}+\omega_x\omega_y\right) \\
&\quad -\Sigma mxy\left(\frac{d\omega_y}{dt}-\omega_z\omega_z\right), \\
M_y &= \Sigma m(z^2+x^2)\frac{d\omega_y}{dt} + \Sigma m(z^2-x^2)\omega_z\omega_x \\
&\quad -\Sigma mzx(\omega_z{}^2-\omega_x{}^2) - \Sigma mxy\left(\frac{d\omega_x}{dt}+\omega_y\omega_z\right) \\
&\quad -\Sigma myz\left(\frac{d\omega_z}{dt}-\omega_x\omega_y\right), \\
M_z &= \Sigma m(x^2+y^2)\frac{d\omega_z}{dt} + \Sigma m(x^2-y^2)\omega_x\omega_y \\
&\quad -\Sigma mxy(\omega_x{}^2-\omega_y{}^2) - \Sigma myz\left(\frac{d\omega_y}{dt}+\omega_z\omega_x\right) \\
&\quad -\Sigma mzx\left(\frac{d\omega_x}{dt}-\omega_y\omega_z\right).
\end{aligned}\right\} \quad . \quad . \quad (25)$$

Thus we have the component moments of the system of forces given in terms of angular accelerations, angular velocities, and mass-coordinate factors of the forms $\Sigma m(x^2+y^2)$ and Σmxy. The form $\Sigma m(x^2-y^2)$ is equal to $\Sigma m(x^2+z^2)-\Sigma m(y^2+z^2)$, and, therefore, need not be considered as a separate form.

If we write $x^2+y^2=r_z{}^2$, r_z being the distance of m from the axis of z, the first of the two general forms of the mass-coordinate factors in (25) is expressed by $\Sigma mr_z{}^2$; or, still more generally, Σmr^2, referring to any right line, r being the distance of m from the line.

7. The Meaning of Σmr^2. The rotational kinetic energy of a body revolving about an axis with the angular velocity ω is $\frac{1}{2}(\Sigma mr^2)\omega^2$, obtained as follows:

The rotational velocity of m at the distance r from the axis is $r\omega$, and its corresponding kinetic energy is $\frac{1}{2}mr^2\omega^2$. Therefore the rotational kinetic energy of the entire body is $\frac{1}{2}(\Sigma mr^2)\omega^2$.

Comparing the expressions for the kinetic energy of translation and of rotation, $\frac{1}{2}Mv^2$ and $\frac{1}{2}(\Sigma mr^2)\omega^2$, *we see that Σmr^2 is associated with the angular velocity about the center of mass in exactly the same way that M is associated with the linear velocity of the center of mass; and* *also that the kinetic energy of rotation about an axis is proportional to the Σmr^2 with respect to that axis, just as the kinetic energy of translation is proportional to the mass of the body.* It may be said, therefore, that the Σmr^2 of a body with respect to an axis measures the capacity of the body to store up kinetic energy of rotation about that axis, in the same sense that the mass of a body measures its capacity to store up kinetic energy of translation.

This is shown also by the rates of change of the energy as functions of v and ω. For translation we have

$$\frac{d(\frac{1}{2}Mv^2)}{dv}=Mv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and for rotation about a given axis (Σmr^2 constant),

$$\frac{d(\frac{1}{2}\Sigma mr^2\omega^2)}{d\omega}=\Sigma mr^2\omega; \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

thus showing that the rate of change of the kinetic energy of translation is proportional to M and that of rotation to Σmr^2.

Although the values of Σmr^2 in (25) refer to axes through the center of mass, the meaning and use of the expression are obviously the same for any axis. *The rotation is about the center of mass in free motion, but in constrained motion the position of the axis depends on the conditions of constraint.*

Σmr^2 defines itself by its notation. It may, however, be described as *the sum of the products obtained by multiplying the mass of each particle by the square of its distance from an axis.*

Σmr^2 is called the moment of inertia of a body with respect to the axis from which r is measured. This interpretation of Σmr^2, according to the method of treating the subject of mechanics in these notes, has no utility; and it is, therefore, preferred that Σmr^2 shall be considered simply as a mass function of the body, and named by reading the symbols employed in designating it. It is thus defined with mathematical accuracy, being simply a mass-distance function which enters the equations of rotation.

The mass-distance functions of the form Σmxy have been called *products of inertia*, but they are completely defined by their mathematical expressions and need no other name.

8. Radius and Center of Gyration. It is often convenient to use an equivalent form of Σmr^2 in which the entire mass of the body shall enter as a factor. If all the particles of a body were at the distance k from an axis we would have

$$\Sigma mr^2=Mk^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and for any body and axis there is a distance

$$k=\sqrt{\frac{\Sigma mr^2}{M}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

at which the entire mass of the body might be concentrated without changing Σmr^2. This distance k is called a *radius of gyration*, and its extemity not on the axis is called a *center of gyration.*

9. Determination of Σmr^2 for Bodies of Geometrical Form. Writing $m=dM=\delta dV$, in which δ is the density of m and V the volume of M, we have

$$\Sigma mr^2=\int r^2\delta dV, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

from which Σmr^2 may be found when $r^2\delta dV$ is integrable.

10. Method of Finding Σmr^2 by Experiment. In discussing the motion of a pendulum it is shown that the time of oscillation of a rigid body about a horizontal axis, under the action of gravity only, is

$$\tau=\pi\sqrt{\frac{(k^2+h^2)}{gh}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

k being the radius of gyration with respect to an axis through the center of mass and parallel to the axis of suspension. This value of τ is closely approximate when the arc of oscillation is small.

Square the above equation for τ, multiply by the mass of the body, and solve for (Mk^2+Mh^2), which is the Σmr^2 of the body with respect to the axis of suspension. We thus have

$$\Sigma mr^2=Mk^2+Mh^2=\frac{Mgh\tau^2}{\pi^2}=\frac{Wh\tau^2}{\pi^2}. \quad . \quad . \quad . \quad (32)$$

Mount the body as a pendulum so that it may oscillate about the given axis, and find the time of oscillation. Measure h if the position of the center of mass be accurately known, or compute it by moments if it cannot be accurately measured. These values of τ and h, together with the weight of the body, give the data for finding Σmr^2 by Eq. (32).

To find h by moments take the axis of suspension as an axis of moments, and attach a dynamometer to the body so that by lifting with the dynamometer the body may be rotated about the axis until h becomes horizontal. The horizontal position of h is secured when the reading of the dynamometer is a maximum. Then Wh is equal to the product of the dynamometer reading by its lever-arm, and this product may be substituted for Wh in (32).

For an axis through the center of mass the experimental method may be used to find the Σmr^2 with respect to an axis of suspension parallel to the given axis. Then the required Σmr^2 is

$$Mk^2 = \frac{Wh\tau^2}{\pi^2} - Mh^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The experimental method is used when the body is not of convenient form for computing Σmr^2, or when varying density makes the computation impracticable.

11. The Σmr^2 Ellipsoid. *For any body and any origin, the locus of points at the distance* $\delta = \frac{1}{\sqrt{\Sigma mr^2}}$ *from the origin is an ellipsoid,* Σmr^2 *being taken with respect to the line upon which* δ *is measured.*

In Fig. 2 let m be any particle of a body, O at any point taken as an origin, and assume $O\delta$ to be laid off equal to the reciprocal of the square root of the Σmr^2 of the body with respect to an axis coincident with $O\delta$. The direction angles of $O\delta$ are α, β, γ, the coordinates of m are x, y, z, and the coordinates of the extremity of δ are x', y', z'. Then we have for the square of $mp = r\delta$,

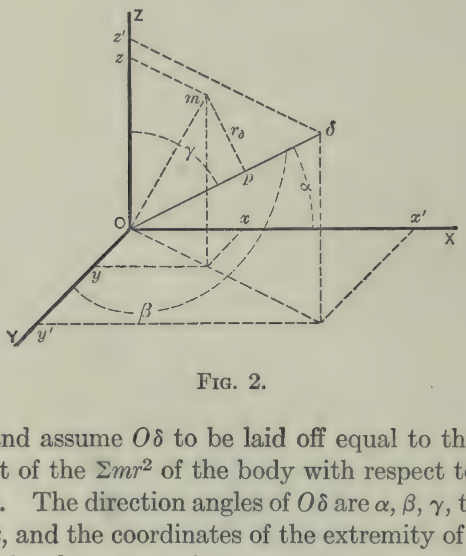

FIG. 2.

$$r_\delta^2 = x^2 + y^2 + z^2 - (x \cos\alpha + y \cos\beta + z \cos\gamma)^2. \quad . \quad . \quad . \quad (34)$$

Expanding and summing the products mr_δ^2 for the entire body we have

$$\begin{aligned} \Sigma mr_\delta^2 = \Sigma mr_x^2 \cos^2\alpha + \Sigma mr_y^2 \cos^2\beta + \Sigma mr_z^2 \cos^2\gamma \\ -2\Sigma mxy \cos\alpha \cos\beta - 2\Sigma mxz \cos\alpha \cos\gamma \\ -2\Sigma myz \cos\beta \cos\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35) \end{aligned}$$

The equation of the locus of the extremity of δ for all axes through O—that is, for all sets of values of α, β, γ—is obtained by eliminating the direction cosines from (35) by means of the following relations:

$$\left.\begin{aligned}\cos\alpha&=\frac{x'}{\delta}=x'\sqrt{\Sigma mr_\delta^2},\\ \cos\beta&=\frac{y'}{\delta}=y'\sqrt{\Sigma mr_\delta^2},\\ \cos\gamma&=\frac{z'}{\delta}=z'\sqrt{\Sigma mr_\delta^2},\end{aligned}\right\}\quad\ldots\ldots\quad(36)$$

Thus we have, after writing A, B, C, for the (Σmr^2)'s with respect to the coordinate axes X, Y, Z, respectively,

$$\begin{aligned}Ax'^2+By'^2+Cz'^2-2(\Sigma mxy)x'y'\\ -2(\Sigma mxz)x'z'-2(\Sigma myz)y'z'=1,\end{aligned}\quad\ldots\quad(37)$$

which is the equation of an ellipsoid.

The equation this ellipsoid referred to its axes of figure is

$$Ax'^2+By'^2+Cz'^2=1;\quad\ldots\ldots\quad(38)$$

that is, Eq. (37) must be reducible to the form (38), and therefore we have the following conditions for the axes of figure of the Σmr^2 ellipsoid:

$$\left.\begin{aligned}\Sigma mxy&=0,\\ \Sigma mxz&=0,\\ \Sigma myz&=0.\end{aligned}\right\}\quad\ldots\ldots\quad(39)$$

12. Principal Axes. The axes of figure of the Σmr^2 ellipsoid are called *principal axes,* and Eqs. (39) are the conditions for principal axes.

13. Principal (Σmr^2)**'s.** The (Σmr^2)'s with respect to principal axes are called *principal* (Σmr^2)'s.

14. Central Ellipsoid. The Σmr^2 at the center of mass is called the *central ellipsoid.*

15. Central Principal Axes. The principal axes at the center mass are called *central principal axes.*

16. The Relation between the Principal (Σmr^2)**'s and the** Σmr^2 **with Respect to any Diameter of the Ellipsoid.** From the general value of Σmr^2, Eq. (35), we have when the coordinate axes are principal,

$$\Sigma mr_\delta^2=A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma.\quad\ldots\quad(40)$$

From this equation we can find the Σmr^2 with respect to any line when the principal (Σmr^2)'s at any point of the line are known and the direction angles of the line with respect to the principal axes are given.

Referring back to Ex. 307 (Probs., U.S.M.A.), the principal axes at the point where any right line pierces the plane of the triangle are the normal and two lines through its foot, since the principal (Σmr^2)'s include the least and the greatest, the greatest being with respect to the normal. Then (40) shows the proposition of Ex. 307 is general.

17. The Relation between the Central Principal (Σmr^2)'s and the Σmr^2 with Respect to Any Line in Space. The distance of the line from the center of mass, d, and the direction angles with reference to the central principal axes, α, β, γ, are assumed to be given. Then, by (40) and Ex. 302,

$$\Sigma mr^2 = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma + Md^2. \quad . \quad . \quad (41)$$

That is, with the given conditions use (40) to find the Σmr^2 with respect to a line through the center of mass and parallel to the given line, and to this result add the product of the mass of the body by the square of the distance of the given line from the center of mass.

18. The Common Name of the Σmr^2 Ellipsoid. The Σmr^2 ellipsoid is commonly called the *momental ellipsoid of inertia*, or, briefly, the *momental ellipsoid*. For reasons already given, the name " Σmr^2 ellipsoid " is preferred.

19. Discussion of the Σmr^2 Ellipsoids of a Body.

1. *To obtain any Σmr^2 from a given ellipsoid.* Having the Σmr^2 ellipsoid of a body at any point, the (Σmr^2)'s of the body with respect to the diameters vary inversely as the squares of the semi-diameters. This is shown by

$$\Sigma mr_\delta{}^2 = \frac{1}{\delta^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

which is the relation established by the conditions introduced in obtaining the equation of the ellipsoid.

In order to obtain the value of any Σmr^2 the scale of the ellipsoid and the units of M and r must, of course, be given. For example, assume M to be given in *gravs* (32 lbs. mass) and r in feet, and take the scale of the ellipsoid to be 1 in.$=\frac{1}{100}$. Then a semi-diameter of 5 in. would represent a Σmr^2 of 400, and this might be the Σmr^2 of a mass of 3220 lb. with a radius of gyration of 2 ft.

Knowing one Σmr^2, then any other can be obtained by the inverse ratio of the square of δ, to any scale.

2. *Limitation as to Form.* The form of the ellipsoid is fixed by the axes of figure. Let these be $2a$, $2b$, $2c$, and assume $a>b>c$. Then Σmr_a^2 is the minimum and Σmr_c^2 the maximum Σmr^2. To determine whether the numerical relations of the semi-axes are limited we have

$$\Sigma mr_a^2 = \Sigma my^2 + \Sigma mz^2, \qquad (43)$$

$$\Sigma mr_b^2 = \Sigma mx^2 + \Sigma mz^2, \qquad (44)$$

$$\Sigma mr_c^2 = \Sigma mx^2 + \Sigma my^2 = \Sigma mr_a^2 + \Sigma mr_b^2 - 2\Sigma mz^2. \qquad (45)$$

That is,

$$\Sigma mr_c^2 < (\Sigma mr_a^2 + \Sigma mr_b^2), \qquad (46)$$

which requires the condition

$$\frac{1}{c^2} < \left(\frac{1}{a^2} + \frac{1}{b^2}\right). \qquad (47)$$

The limit of this relation is $\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}$, which is the relation for the hypothetical plane figure.

3. *Special Cases.* When $a=b$ the ellipsoid is an oblate spheroid, and when $b=c$ the spheroid is prolate. In both cases all axes in the plane of the equator are principal and their (Σmr^2)'s equal.

When the $a=b=c$ ellipsoid is a sphere, and all axes are principal and their (Σmr^2)'s equal.

The limit of form for the oblate spheroid is $\frac{1}{c^2} < \frac{2}{a^2}$, or $a < c\sqrt{2}$; that is, a lies between c and $c\sqrt{2}$. For the prolate spheroid $\frac{1}{c^2} < \frac{1}{a^2} + \frac{1}{c^2}$, and a lies between c and ∞.

For the oblate spheroid a approaches $c\sqrt{2}$ as the form of the body approaches that of a plane figure, and for the prolate spheroid a approaches ∞ as the form of the body approaches that of a right line.

4. *In passing to ∞ along the right line containing any radius vector, δ, of the central ellipsoid, the Σmr^2 ellipsoid diminishes to a right line of length 2δ.* For the ellipsoid at any point on the assumed line, the radius vector coincident with the line is the same as for the central ellipsoid, since its length is determined from the Σmr^2 with respect to the same line; while any section of the ellipsoid normal to the line diminishes as the point recedes from the center of mass, and vanishes at infinity.

See Figs. Ex. 312 to 315.

20. Euler's Equations of Rotation. The second reduction of Eqs. (R_{cm}) is made by referring the equations

$$\left.\begin{aligned}
M_x&=\Sigma m(y^2+z^2)\frac{d\omega_x}{dt}+\Sigma m(y^2-z^2)\,\omega_y\omega_z\\
&\quad-\Sigma myz(\omega_y{}^2-\omega_z{}^2)-\Sigma mzx\left(\frac{d\omega_z}{dt}+\omega_x\omega_y\right)\\
&\quad-\Sigma mxy\left(\frac{d\omega_y}{dt}-\omega_z\omega_x\right),\\
M_y&=\Sigma m(z^2+x^2)\frac{d\omega_y}{dt}+\Sigma m(z^2-x^2)\,\omega_z\omega_x\\
&\quad-\Sigma mzx(\omega_z{}^2-\omega_x{}^2)-\Sigma mxy\left(\frac{d\omega_x}{dt}+\omega_y\omega_z\right)\\
&\quad-\Sigma myz\left(\frac{d\omega_z}{dt}-\omega_x\omega_y\right),\\
M_z&=\Sigma m(x^2+y^2)\frac{d\omega_z}{dt}+\Sigma m(x^2-y^2)\,\omega_x\omega_y\\
&\quad-\Sigma mxy(\omega_x{}^2-\omega_y{}^2)-\Sigma myz\left(\frac{d\omega_y}{dt}+\omega_z\omega_x\right)\\
&\quad-\Sigma mzx\left(\frac{d\omega_x}{dt}-\omega_y\omega_z\right),
\end{aligned}\right\}\quad . \quad . \quad . \quad (25)$$

to principal axes of the body as coordinate axes. Retaining the notation A, B, C for the principal (Σmr^2)'s and applying the conditions for principal axes, (39), we thus have

$$\left.\begin{aligned}
M_x&=\frac{A\,d\omega_x}{dt}-(B-C)\,\omega_y\omega_z,\\
M_y&=\frac{B\,d\omega_y}{dt}-(C-A)\,\omega_x\omega_z,\\
M_z&=\frac{C\,d\omega_z}{dt}-(A-B)\,\omega_x\omega_y.
\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (48)$$

These are *Euler's equations of rotation,* in which the component moments of the system of forces are given in terms of the principal (Σmr^2)'s and the component angular accelerations and velocities with respect to the principal axes. Having been deduced from (R_{cm}) they apply to rotation about a fixed axis or a fixed point, as well as to free rotation about the central principal axes.

21. Free Rotation about a Single Central Principal Axis and Constrained Rotation about a Fixed Axis. These are the simplest cases in the application of Euler's equations. For both we have taking X to be the axis of rotation and solving for the angluar acceleration,

$$\frac{d\omega_x}{dt}=\frac{M_x}{\Sigma m{r_x}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

since ω_y and ω_z are both zero. Or, in more general terms, we have

$$\frac{d^2\theta}{dt^2}=\frac{Rl}{\Sigma mr^2}=\frac{\mathbf{Rl}}{(\mathbf{Mk}^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Rl being the moment of R with respect to the axis of rotation.

This relation, (50), may also be deduced by direct analysis when the rotation is wholly about a single axis. The effect of R is transmitted to the rotating solid so that each particle is acted on by a force $m\left(\frac{rd^2\theta}{dt^2}\right)$, tangent to the arc in which m rotates. The moment of this tangential force is $mr^2\left(\frac{d^2\theta}{dt^2}\right)$, and the sum of the moments transmitted to all of the particles is $\Sigma mr^2\left(\frac{d^2\theta}{dt^2}\right)$. This aggregate must be equal to the applied moment, Rl, and solving the equality for the angular acceleration we have (50).

The centripetal force component of R has zero moment with respect to the axis. Comparing $\frac{d\omega}{dt}=\frac{Rl}{\Sigma mr^2}$ with $\frac{dv}{dt}=\frac{R}{M}$, we see the exact parallelism of the two relations; the angular acceleration, the moment of the resultant, and the Σmr^2 having the same relation as the linear acceleration of the center of mass, the resultant and the mass of the body. This parallelism becomes still more striking when we substitute Mk^2 for Σmr^2 and assume $l=k$. We thus get $\frac{kd\omega}{dt}=\frac{R}{M}$. This is equivalent to assuming an indefinitely thin hollow right cylinder with circular base, with R applied tangent to the cylinder and normal to the axis. All of the particles would then have the same tangential acceleration in their rotational motion, and this acceleration is numerically equal to the linear acceleration of the center of mass in the free motion of M under the action of R.

Equation (50) may be verified and illustrated by means of a cylinder mounted on an axis and attached to Atwood's machine, as follows:

22. Rotation Attachment to Atwood's Machine. The attachment is shown in Fig. 3. A solid steel cylinder is mounted so that it can be rotated by a falling weight attached to the end of a fine silk thread wound on the cylinder. The time of fall through a known height is recorded, and the acceleration of the falling weight is obtained from $A=\frac{2z}{t^2}$. This result should, of course, agree with the solution of the problem of motion under the known conditions.

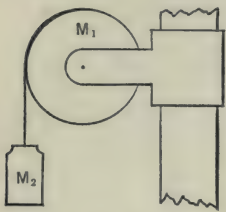

FIG. 3.

Let M_1 be the mass of the cylinder and M_2 the mass of the falling weight, both in grams; ρ the radius of the cylinder; l the lever-arm of the weight, and M_f the moment of the friction.

The resultant force in the line of the thread is $\left(M_2g-\frac{M_f}{l}\right)$ dynes, and the force which accelerates the cylinder is $\left(M_2g-\frac{M_f}{l}\right)-M_2l\frac{d^2\theta}{dt^2}$, since the force $M_2l\frac{d^2\theta}{dt^2}$ is required to accelerate M_2. Then we have according to (50),

$$\frac{d^2\theta}{dt^2}=\left(M_2g-\frac{M_f}{l}-M_2l\frac{d^2\theta}{dt^2}\right)\frac{l}{(M_1k^2)}. \quad . \quad . \quad . \quad (51)$$

Solving (51) for the angular acceleration we have

$$\frac{d^2\theta}{dt^2}=\frac{(M_2gl-M_f)}{(M_2l^2+M_1k^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

This form may be obtained more directly by supposing M_2 to be distributed uniformly over the surface of the cylinder, and then assuming the cylinder thus increased to rotate under the moment M_2gl-M_f; thus showing (52) to be an equivalent of (51), since each particle of M_2 would have the same linear acceleration in both cases.

23. The Instantaneous Axis. In the foregoing solutions of problems in rotation the axes are taken as permanent, and the simple relation given in Eq. (50) governs the angular acceleration. In the general case, however, there is a component angular velocity about each of the principal axes, and these components vary with the time. Therefore the resultant axis changes its position in the body in accordance with the changes in the component angular velocities. These changes take place by continuity, and, therefore, the resultant axis describes a conical surface whose apex is at the center of rotation. For example, the familiar wabbling motion of a football, when the moment axis of the kick is oblique to the axis of the ball, is due to a continuous change in the resultant axis of rotation.

In Fig. 4 the ellipse may be taken as a section of the ball (or of its Σmr^2 ellipsoid). The resultant angular velocity about the instantaneous axis has the principal components ω_x and ω_z, and the wabbling component $\omega \cos \phi$ about the axis of the kick.

The resultant axis generally occupies a single position for an instant only, and it is, therefore, called the *instantaneous axis;* likewise the resultant angular velocity is called the *instantaneous angular velocity.*

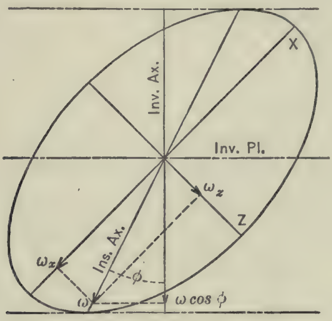

FIG. 4.

The equations of the instantaneous axis are obtained as follows: From Eqs. (13) we have for the component linear velocities of any particle of the body with respect to the center of rotation, omitting accents,

$$\left.\begin{aligned}\frac{dx}{dt}&=z\omega_y-y\omega_z,\\ \frac{dy}{dt}&=x\omega_z-z\omega_x,\\ \frac{dz}{dt}&=y\omega_x-x\omega_y.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Any point on the instantaneous axis has no motion with respect to the center of rotation, and therefore we obtain the equations of the locus of such points—that is, of the instantaneous axis—by making the component linear velocities zero in Eq. (53). We thus have

$$\left.\begin{aligned} z\omega_y - y\omega_z &= 0, \\ x\omega_z - z\omega_x &= 0, \\ y\omega_x - x\omega_y &= 0. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (54)$$

Any one of these equations is dependent on the other two, and for each set of values of the component angular velocities the equations determine a right line which is the resultant axis at the instant for which the set of values is taken. We therefore see that the instantaneous axis describes a cone while the component angular velocities change with the time, the rectilinear elements of the cone passing through the center of rotation.

The direction cosines of the axis are $\frac{\omega_x}{\omega}$, $\frac{\omega_y}{\omega}$, and $\frac{\omega_z}{\omega}$.

23a. The Spontaneous Axis. It may be defined as a *line at rest about which a free body rotates during impact.* To find its equations and the conditions for its development we proceed as follows:

By dividing Eqs. (14) by dt, and changing the notation for the component velocities of the center of mass and the angular velocities about the movable axes, we have

$$\left.\begin{aligned} \frac{dx_0}{dt} &= v_x + z'\omega_y - y'\omega_z, \\ \frac{dy_0}{dt} &= v_y + x'\omega_z - z'\omega_x, \\ \frac{dz_0}{dt} &= v_z + y'\omega_x - x'\omega_y, \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

for the component linear velocities of the particle x', y', z', with respect to the fixed origin; v_x, v_y, and v_z, being the component linear velocities imparted to the center of mass by the impact.

The spontaneous axis being at rest we have $\frac{dx_0}{dt} = \frac{dy_0}{dt} = \frac{dz_0}{dt} = 0$ for all of its points, and therefore we get its equations by introducing these conditions in (1), par. 23*a*. We thus have, omitting accents.

$$\left.\begin{aligned} v_x + z\omega_y - y\omega_z &= 0, \\ v_y + x\omega_z - z\omega_x &= 0, \\ v_z + y\omega_x - x\omega_y &= 0, \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

which show that the spontaneous axis is parallel to the instantaneous axis, whose equations are (54).

Eqs. (2), par. 23 are dependent on the condition

$$v_x\omega_x+v_y\omega_y+v_z\omega_z=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

obtained by multiplying each by the component angular velocity which it does not contain, and then adding the resulting equations.

Dividing (3), par 23*a*, by $v\omega$ we have

$$\frac{v_x}{v}\frac{\omega_x}{\omega}+\frac{v_y}{v}\frac{\omega_y}{\omega}+\frac{v_z}{v}\frac{\omega_z}{\omega}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which shows that the line of impact (parallel to v) is perpendicular to the instantaneous axis.

This condition is fulfilled whenever the line of impact (in the invariable plane) is normal to the plane of the invariable and instantaneous axes. The general case is shown in Fig. 5. The instantaneous axis is taken as the axis of z, and the axis of y and the line of impact are normal to the plane of the figure, the point of impact being at I. Then we have, M being the mass of the body and v the velocity of its center of mass,

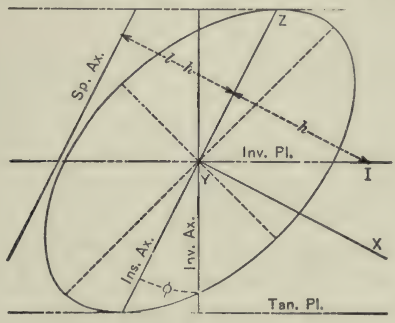

FIG. 5.

$$\left.\begin{aligned} &v_x=v_z=\omega_z=\omega_y=0,\\ &v_y=v,\\ &\omega_z=\omega=\frac{Mvh}{Mk^2}=\frac{vh}{k^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These values reduce the equations of the spontaneous axis to

$$\left.\begin{aligned} &x=-\frac{v_y}{\omega_z}=-\frac{k^2}{h},\\ &y=0,\\ &z=\frac{0}{0}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We therefore see that the spontaneous axis is on the opposite side of the instantaneous axis from the line of impact, at a distance equal to the square of the radius of gyration divided by the lever-arm of the impact, both taken with respect to the instantaneous axis.

From the relation between the length of the compound pendulum and its equivalent simple pendulum, h being the distance to the c.g., we have

$$(l-h)=\frac{k^2}{h}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and comparing this with the value of x in (6, par. $23a$) we see that the relation of distances between the line of impact and the instantaneous and spontaneous axes is the same as for the centers of oscillation and mass and the axis of suspension.

The line l passes through the center of mass when the instantaneous axis is principal.

24. Rotation of a Rigid Solid under no Extraneous Forces. It is assumed in this case that the body has been given a motion of rotation and then abandoned to itself without any extraneous force continuing to act upon it. The football kicked into the air is a good illustration, neglecting the effect of the air. The character of the motion is modified by the action of the air, but not sufficiently to obscure the typical motion about a continually changing instantaneous axis.

There being no extraneous forces acting on the body, the moments as expressed in Eqs. (R_{cm}) are zero, and we, therefore, have

$$\left.\begin{aligned}\Sigma m\left(\frac{yd^2z}{dt^2}-\frac{zd^2y}{dt^2}\right)&=0,\\ \Sigma m\left(\frac{zd^2x}{dt^2}-\frac{xd^2z}{dt^2}\right)&=0,\\ \Sigma m\left(\frac{xd^2y}{dt^2}-\frac{yd^2x}{dt^2}\right)&=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (55)$$

From these we get by integration,

$$\left.\begin{aligned}\Sigma m\left(\frac{ydz}{dt}-\frac{zdy}{dt}\right)&=C_x,\\ \Sigma m\left(\frac{zdx}{dt}-\frac{xdz}{dt}\right)&=C_y,\\ \Sigma m\left(\frac{xdy}{dt}-\frac{ydx}{dt}\right)&=C_z.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (56)$$

25. The Conservation of Moments. The constants C_x, C_y, C_z, in (56) may be interpreted by writing the parts belonging to m in the form $mv_zy - mv_yz$, which is the moment of the momentum of m with respect to the axis of x. Eqs. (56), therefore, express the constancy of the moments of the momentum of the body with respect to any set of rectangular coordinate axes fixed in direction through the center of rotation.

This principle is true for any axis through the center of rotation, since any such axis may be taken as one of the coordinate axes in (56). It is called *the principle of the conservation of moments,* and may be stated as follows:

When a rigid body is rotating about a center under no extraneous forces the moment of the momentum with respect to any axis through the center of rotation is constant.

Eqs. (56) are true separately, and, therefore, the principle of the conservation of moments is also true for any axis with respect to which the moment of the applied forces is zero.

26. The Invariable Axis. The resultant moment of the momentum of the body is given by

$$C_R = \sqrt{C_x^2 + C_y^2 + C_z^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

and the axis of C_R is called the *invariable axis.*

The invariable axis is marked *Inv. Ax.* in Fig. 58.

27. The Invariable Plane. The plane through the center of rotation and normal to the invariable axis is called the *invariable plane.*

The Σmr^2 ellipsoid at the center of rotation has a fixed relation to the body, and therefore the motion of one determines that of the other; and since the invariable axis and invariable plane are fixed in direction, we may use them as means of reference in tracing the motion of the ellipsoid about the center of rotation.

28. The Principal Moments of Momentum. By substituting for the linear velocities in (56) their values as given in (53) we have

$$\left.\begin{aligned} C_x &= \Sigma m(y^2+z^2)\omega_x - \Sigma mxy\omega_y - \Sigma mxz\omega_z, \\ C_y &= \Sigma m(x^2+z^2)\omega_y - \Sigma myz\omega_z - \Sigma myx\omega_x, \\ C_z &= \Sigma m(x^2+y^2)\omega_z - \Sigma mzx\omega_x - \Sigma mzy\omega_y; \end{aligned}\right\} \quad . \quad . \quad . \quad (58)$$

and when the axes are principal these become

$$\left.\begin{aligned} C_x &= A\omega_x, \\ C_y &= B\omega_y, \\ C_z &= C\omega_z, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

which may be called the principal moments of momentum.

From these we have $\omega_x = \frac{C_x}{A}$, $\omega_y = \frac{C_y}{B}$, $\omega_z = \frac{C_z}{C}$. That is, *when a rigid body is moving without extraneous forces acting upon it, the angular velocity about a central principal axis is equal to the moment of the momentum divided by the* Σmr^2 *with respect to that axis.*

This principle is also true for constrained motion about principal axes through a fixed point under no extraneous forces.

It is true for the instantaneous axis, for, if one of the coordinate axes in (58) be the instantaneous axis then the component angular velocities about the other axes are zero, and we have the same relation as in (59) for $\Sigma m(r\omega)r$, Σmr^2, and ω, $\Sigma m(r\omega)r$ being the moment of the momentum with respect to the instantaneous axis. The relation is obvious in this case, since it may be stated directly from the given expressions: that is, $\Sigma m(r\omega)r = \Sigma mr^2\omega$; or $\omega = \frac{\Sigma m(r\omega)r}{\Sigma mr^2}$.

29. The Invariable Angular Velocity, and the Law of Variation of the Resultant Angular Velocity. Let the Σmr^2 with respect to the instantaneous axis be indicated by Σmr_i^2, and let ϕ be the angle between the invariable and the instantaneous axis. Then the component moment of the momentum with respect to the instantaneous axis is $C_R \cos\phi$, and, therefore, we have for the instantaneous angular velocity,

$$\omega = \frac{C_R \cos\phi}{\Sigma mr_i^2}; \quad . \ . \ . \ . \ . \ . \ . \ . \quad (60)$$

and for the component angular velocity about the invariable axis, from (60), multiplying by ω and solving for $\omega \cos\phi$,

$$\omega \cos\phi = \frac{\omega^2 \Sigma mr_i^2}{C_R} = \text{a constant}. \quad . \ . \ . \ . \quad (61)$$

The constancy of this value is shown by the fact that the kinetic energy of the body is constant, there being no forces acting to change the energy. (This fact can be shown analytically by applying Euler's equations, multiplying each by the angular velocity which it does not contain, then adding and integrating.)

From (61) we see that the angular velocity about the invariable axis is equal to twice the kinetic energy divided by the resultant moment of the momentum; and that the resultant angular velocity varies inversely as the direction cosine of the instantaneous axis.

Taking δ to be the semidiameter of the ellipsoid coincident with the instantaneous axis, we have, multiplying (42) by ω^2 and taking the square root,

$$\frac{\omega}{\delta}=\sqrt{\omega^2\Sigma m r_i^2}=\text{a constant}, \quad . \quad . \quad . \quad . \quad . \quad (62)$$

and, therefore, ω varies directly with δ.

30. The Polhode and the Herpolhode. For the projection of δ on the invariable axis we have from (62),

$$\delta \cos \phi=\frac{\omega \cos \phi}{\sqrt{\omega^2\Sigma m r_i^2}}=\text{a constant}. \quad . \quad . \quad . \quad . \quad (63)$$

That is, the extremity of δ on the ellipsoid, which we may call *the instantaneous pole*, lies in a plane which is at a constant distance from the invariable plane.

The locus of the instantaneous pole on the ellipsoid is called *the polhode*, and its locus in the fixed plane at the distance $\delta \cos \phi$ from the invariable plane is called *the herpolhode.*

31. The Relation of the Ellipsoid to the Plane of the Herpolhode. To find the relation of the plane of the herpolhode to the ellipsoid, we may write its equation referred to the principal axes,

$$\left(\frac{C_x}{C_R}\right)x+\left(\frac{C_y}{C_R}\right)y+\left(\frac{C_z}{C_R}\right)z=\delta \cos \phi, \quad . \quad . \quad . \quad . \quad (64)$$

since we know the equation of the invariable plane to be

$$\left(\frac{C_x}{C_R}\right)x+\left(\frac{C_y}{C_R}\right)y+\left(\frac{C_z}{C_R}\right)=0. \quad . \quad . \quad . \quad . \quad . \quad (65)$$

To introduce the coordinates of the instantaneous pole into the equation of the plane of the herpolhode, (64) let these coordinates be x', y', z', and write

$$C_x=A\omega_x=A\omega\left(\frac{x'}{\delta}\right)=Ax'\left(\frac{\omega}{\delta}\right)=Ax'\delta \cos \phi \, C_R,$$

the last step by (62) and (60); so also,

$$C_y=By'\delta \cos \phi \, C_R,$$

$$C_z=Cz'\delta \cos \phi \, C_R. \quad . \quad (66)$$

These in (64) give

$$Axx'+Byy'+Czz'=1, \quad . \quad . \quad . \quad . \quad . \quad (67)$$

which is the equation of the tangent plane to the ellipsoid at the instantaneous pole. Therefore, *the ellipsoid is constantly tangent to the plane of the herpolhode.* This plane is marked *Tan Pl.* in Fig. 59.

32. General Character of the Motion. Fig. 6 represents the XY section of the ellipsoid projected on the vertical plane, $a > b > c$, with the herpolhode shown in the horizontal plane between and tangent to two limiting circumferences. The polhode is projected in *abc.* It is in general a closed curve on the ellipsoid, and rolls into contact with the tangent plane along the herpolhode. During this motion the instantaneous axis generates two cones having a common vertex at the center of the ellipsoid; one a fixed cone whose base is the herpolhode, and the other a movable cone whose directrix is the polhode. The first is fixed with respect to the center of rotation, and the second is fixed with respect to the ellipsoid but movable about the center of rotation. These two cones are tangent to each other along the instantaneous axis, which is a common element of both; and, therefore, the cone fixed in the ellipsoid rolls on the cone fixed in space. The one fixed in the ellipsoid is called the *rolling cone* and the other the *directing cone.*

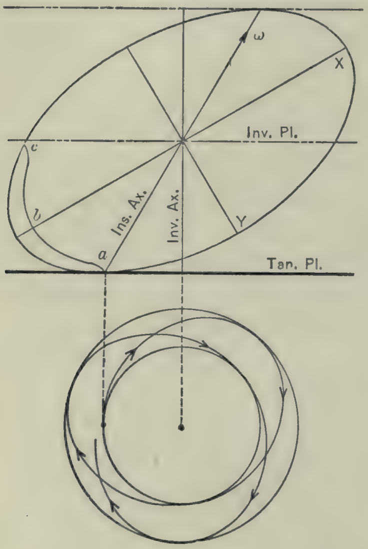

Fig. 6.

The herpolhode may be regarded as the curve into which the polhode develops on the tangent plane while the rolling cone moves over the directing cone. In general it consists of a number of intersecting convolutions, and is a closed curve when $(\omega \cos \phi)\tau$ is commensurable with π, τ being the time required for the rolling cone to make one complete turn on the directing cone.

It is obvious that if the ellipsoid be a prolate spheroid, as in the case of the football, the polhode and herpolhode will both be circular arcs, since ϕ is then constant.

The different cases which arise depend on the form of the rolling cone, and the details of the motion may be obtained from a discussion of its equation.

33. Equation of the Rolling Cone. To get the equation of the rolling cone we simply form a homogeneous equation of the second degree in the coordinates of the instantaneous pole. This may be done by equating two equal functions of these coordinates, as follows:

One of the functions is given by the equation of the ellipsoid, since the instantaneous pole is always on the ellipsoid. We thus have

$$Ax'^2+By'^2+Cz'^2=1. \quad . \quad . \quad . \quad . \quad . \quad (68)$$

The other is obtained by taking the sum of the squares of the direction cosines of the invariable axis, $\left(\frac{C_x}{C_R}\right)^2+\left(\frac{C_y}{C_R}\right)^2+\left(\frac{C_z}{C_R}\right)^2=1$, and reducing to the form

$$\delta^2 \cos^2 \phi(A^2x'^2+B^2y'^2+C^2z'^2)=1, \quad . \quad . \quad . \quad (69)$$

by Eq. (66).

Subtracting (69) from (68) we have, writing p for $\delta \cos \phi$ and omitting accents,

$$A(1-p^2A)x^2+B(1-p^2B)y^2+C(1-p^2C)z^2=0, \quad . \quad . \quad (70)$$

which is the equation of the rolling cone.

It is readily seen that this is the equation of the locus of the instantaneous axis in the ellipsoid, since it is satisfied by the coordinates of the center, by the coordinates of the polhode, and by any fraction of any set of the coordinates of the polhode.

A more convenient form for discussion is obtained by substituting for the (Σmr^2)'s in parenthesis their values in terms of the semi-axes of the ellipsoid. We thus have

$$A\left(\frac{1-p^2}{a^2}\right)x^2+B\left(\frac{1-p^2}{b^2}\right)y^2+C\left(\frac{1-p^2}{c^2}\right)z^2=0. \quad . \quad . \quad (71)$$

34. Discussion of the Rolling Cone. The normal distance, p, from the center to the tangent plane, may have any value from a to c.

1. Let $p=a$. Then Eq. (71) becomes

$$B\left(\frac{1-p^2}{b^2}\right)y^2+C\left(\frac{1-p^2}{c^2}\right)z^2=0, \quad . \quad . \quad . \quad . \quad (72)$$

which can be satisfied only when $y=z=0$. The rolling cone is then given by $x=\frac{0}{0}$, $y=0$, $z=0$. That is, the rolling cone, the directing cone, the invariable axis, and the instantaneous axis all coincide with the longest principal axis, and the polhode and herpolhode are coincident points at the extremity of this axis.

2. Let $a>p>b$. Then the first term in the equation of the cone is positive and the others are negative. X is the axis of the cone and the sections of the cone normal to X are ellipses.

The polhode is a closed curve of double curvature, and the herpolhode lies between two circumferences whose common center is the point where the invariable axis pierces the tangent plane. A complete analysis shows that the herpolhode is concave toward the center, as in Fig. 6, and that it is tangent to the outer and inner circles when the radius vector, $\delta \sin \phi$, has its maximum and minimum values, respectively.

3. Let $p=b$. Then we have from the equation of the rolling cone

$$A\left(\frac{1-b^2}{a^2}\right)x^2-C\left(\frac{b^2}{c^2-1}\right)z^2=0, \quad . \quad . \quad . \quad . \quad . \qquad (73)$$

giving two planes whose equations are

$$y=\tfrac{0}{0}, \quad x=\pm\left(\frac{a^2}{c^2}\right)\sqrt{\frac{(b^2-c^2)}{a^2-b^2}}\,z. \quad . \quad . \quad . \quad . \quad . \qquad (74)$$

These planes are equally inclined to XY and intersect in Y. They cut two equal ellipses from the ellipsoid, and these are called the *critical ellipses*, or *separating polhodes*, since they divide the ellipsoid into two areas which contain all of the polhodes belonging to the longest and shortest axes; or into four areas, two for each axis, if we consider both nappes of the rolling cones. The conditions of motion are duplicated on opposite sides of the invariable plane, and in the discussion it is necessary to consider only one of the symmetrical pair of tangent planes.

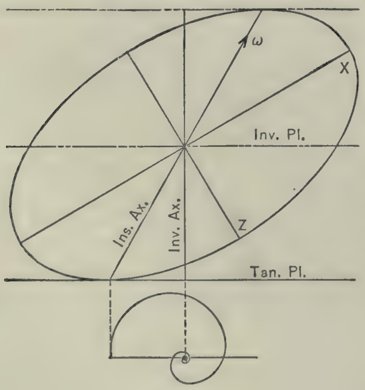

FIG. 7.

Fig. 7 shows this case. The instantaneous axis is in the plane of one of the critical ellipses, and the elliptical polhode rolls into a spiral herpolhode whose maximum radius vector corresponds to the maximum semidiameter of the critical ellipse as an instantaneous axis, as in the figure; and the other limiting radius vector of the spiral is zero, which occurs when the mean semi-axis of the ellipsoid coincides with the invariable axis.

The ellipsoid may roll in either direction from the position which gives the maximum radius vector, depending on the sign of $\omega \cos \phi$, and the herpolhode is, therefore, a symmetrical double spiral. (Only one-half is shown in the figure.) The equation of this spiral is *

$$\rho = \frac{2bn}{(e^{n\theta} + e^{-n\theta})}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (75)$$

in which $n = \sqrt{\frac{(C-B)(B-A)}{(AC)}}$ and θ is measured from the maximum value of ρ. If the instantaneous axis coincide initially with b it will remain so, and the rotation will continue uniformly about this axis; but if the initial position of the instantaneous axis be in the plane of one of the critical ellipses and not coincident with b, then

---

* Omit the accents from (68) and (69), substitute $\frac{1}{B}$ for $\delta^2 \cos^2 \phi$, and denote the radius vector of the herpolhode by ρ. We then have the following relations between the coordinates of the instantaneous pole:

$$\left.\begin{aligned} Ax^2 + By^2 + Cz^2 &= 1, \\ A^2x^2 + B^2y^2 + C^2z^2 &= B, \\ x^2 + y^2 + z^2 &= \frac{\rho^2 + 1}{B}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

From these

$$\left.\begin{aligned} x^2 &= \frac{BC\rho^2}{(B-A)(C-A)}, \\ y^2 &= \frac{C + A - B - AC\left(\frac{\rho^2+1}{B}\right)}{(C-A)(B-A)}, \\ z^2 &= \frac{AB\rho^2}{(C-A)(C-B)}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

The differential of the arc of the herpolhode is equal to the differential of the arc of the polhode, since the herpolhode is generated by the rolling of the polhode on the tangent plane. Let θ be the angle which ρ makes with the prime radius vector, and we have from the equality of the differential arcs,

$$d\rho^2 + \rho^2 d\theta^2 = dx^2 + dy^2 + dz^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (c)$$

By differentiating (b) and substituting in (c) we obtain

$$\rho^2 d\theta^2 = \frac{ACd\rho^2}{(C-B)(B-A) - ABC\rho^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (d)$$

Then by writing $(C-B)(B-A) = ACn^2$ and solving for $d\theta$ we have

$$d\theta = \frac{d\rho}{\rho\sqrt{n^2 - B\rho^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (e)$$

whose integration gives (75).

it will require an infinite value of θ to reach b. That is, the zero value of ρ becomes impossible, and b is a limit which the instantaneous axis approaches but never reaches.

Ex. 343. Assume the central ellipsoid to be given by $a=6$, $b=5$, $c=4$, and construct the herpolhode for $\delta \cos \phi=b$, from $\theta=0$ to $\theta=4\pi$. Also compute ρ for 100π.

4. Let $b>p>c$. The discussion is similar to that for $a>p>b$, the axis of the rolling cone being the shortest axis of the ellipsoid.

5. Let $p=c$. The discussion is similar to that for $p=a$.

35. Permanency of Rotation. We have seen that the instantaneous axis coincides permanently with the invariable axis whenever the resultant rotation is about any one of the axes of figure of the central ellipsoid. For this reason the axes of figure are called *permanent axes.*

In the case of either of the spheroids the rotation is permanent about any axis in the plane of the equator, as well as about the polar axis.

36. Stability of Rotation. There is no axis of the ellipsoid which is stable in the sense that there is an immediate return to rotation about that axis after a small displacement due to an increment of angular momentum. However, if such an increment should slightly displace the instantaneous axis from a or c, this would establish a small polhode surrounding the end of a or c, and the rotation would then continue about an instantaneous axis near a or c if the body be subjected to no further disturbance. In this sense a and c are called *stable axes.*

With regard to rotation about the mean axis, an impulse that would slightly displace the instantaneous axis from b would almost certainly throw it into a polhode belonging to a or c. The rotation would then carry the instantaneous pole throughout this new polhode, and there would be no element of stability in the relation of the instantaneous axis to the mean axis of the ellipsoid. The mean axis is, therefore, called an *unstable axis.*

There is, however, a very limited sense in which the mean axis is stable to even a greater degree than the two other axes. If we assume an impulse which would throw the instantaneous axis from b into the plane of a critical ellipse, then after this impulse the instantaneous axis would approach b either directly or through the maximum radius vector, depending on the sign of ω, and in a few turns around the invariable axis the instantaneous pole would be describing the spiral herpolhode very near the invariable axis. This condition can exist for displacements of the instantaneous pole from b in only two of the infinite number of planes containing b; that is, in the planes of the critical ellipses.

37. Rotation of a Rigid Solid about a Center under the Action of Force. In Euler's equations of rotation (48), the reference is to principal axes, and these are fixed with respect to the body. They therefore conform to the motion of the body, thus becoming movable axes whose motions must be determined with respect to fixed means of reference, in order to solve the problem of motion. Euler's equations are applicable to both free rotation about the center of mass and to constrained rotation about a fixed center, and for the motion of rotation the reference in either case is to a rectangular set of axes through the center of rotation and fixed in direction. These axes may be considered as fixed in both cases, since the translation of the center of mass has no effect on the rotation of the body about that center.

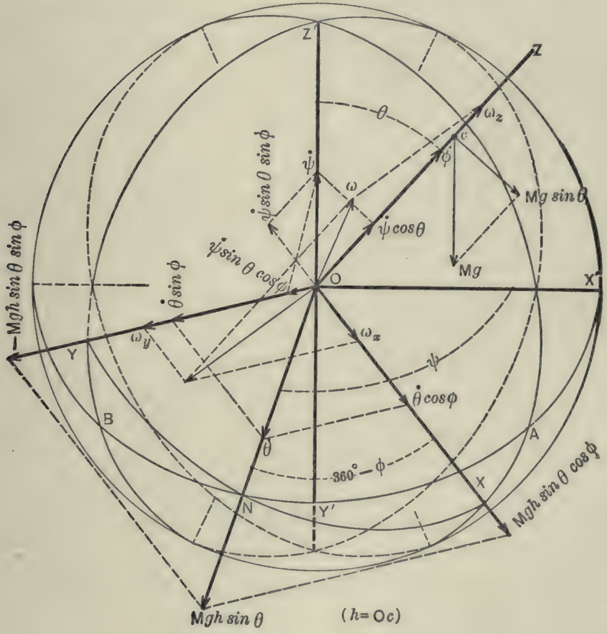

FIG. 8.

To illustrate the method of relating the principal to the fixed axes, assume a sphere of unit radius described about the center of rotation, as in Fig. 8. The fixed axes are X', Y', Z', the eye being in the plane $Y'Z'$ and above $X'Y'$. The principal axes, X, Y, Z, are related to the fixed axes by the angles θ, ψ, ϕ; θ from Z' to Z, equal to ANX; ψ from X' to ON; and ϕ from ON to X. The angle θ is called the *obliquity*, and ψ the *precession*.

The auxiliary angular velocities $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$ and $\frac{d\phi}{dt}$ are used to define the motion of the principal axes of the body with respect to the coordinate axes fixed in direction, and the complete solution of the problem requires the integration of the differential equations derived from the values of these auxiliary angular velocities. Thus the angles θ, ψ, and ϕ become known in terms of the time.

In illustrating the relation between the auxiliary and principal angular velocities in Fig. 61, the former are indicated by $\dot\theta$, $\dot\psi$, and $\dot\phi$. The angular velocity $\dot\theta$ is the *nutation of obliquity;* that is, the *nodding* of the Z axis of the body to and from the fixed axis Z'; while $\dot\psi$ is the *precessional velocity*, giving the rate of motion of the intersection ON in the fixed plane $X'Y'$, equal to and 90° from the rotation of the azimuth plane through Z' and Z. This motion is *direct* when $\dot\psi$ is positive and *retrograde* when negative. The motion of X with respect to ON is given by $\dot\phi$.

By projecting the auxiliary angular velocities on the principal axes we have, as shown in the figure,

$$\left.\begin{aligned}\omega_x&=\dot\theta\cos\phi+\sin\dot\psi\,\theta\sin\phi,\\ \omega_y&=+\dot\theta\sin\phi+\dot\psi\sin\theta\cos\phi,\\ \omega_z&=\dot\phi+\dot\psi\cos\theta.\end{aligned}\right\}\quad\ldots\ldots\quad(76)$$

(From the spherical triangles ANX and BNY we have $\cos Z'OX=\sin XOA=\sin XON\sin\theta=\sin\theta\sin\phi$: $\cos Z'OY=\sin YOB=\sin YON\sin\theta=\sin\theta\cos\phi$. Also $NOY=\phi+90°$ and $\cos NOY=-\sin\phi$.)

These equations (76), *together with Euler's equations of rotation* (48), *furnish the means of solving the general problem of rotation under the action of force.*

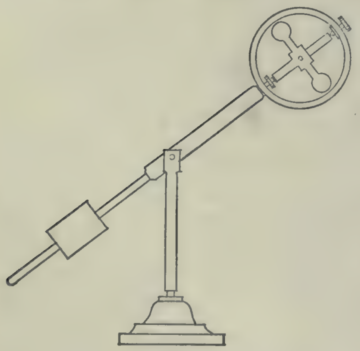

FIG. 9.

Ex. 344. **The Gyroscope.** Assume the common form of gyroscope in which the instrument is constrained to rotate about a point on the axis of the wheel such that the weight shall have a moment with respect to the center of rotation, as indicated in Fig. 9. The distance of the center of gravity from the center of rotation is regulated by moving the adjustable weight on the rod. The rotating wheel is shown in section. In general the motion consists of the rotation of the wheel, ω_z; the dipping of the wheel and its frame about a horizontal axis, θ; and the precession about the vertical axis, $\dot{\psi}$.

1. Spin the wheel of the gyroscope, and then let the axis move from rest under the action of gravity. Explain the general character of the motion.

Take the axis of the wheel as the principal axis Z, and in Fig. 8 let the point c be the center of gravity. The weight is indicated by Mg, and its moment with respect to ON is $Mgh \sin \theta$, h being the distance of the center of gravity from the center of rotation. The component moments with respect to X and Y are $Mgh \sin \theta \cos \phi$, and $-Mgh \sin \theta \sin \phi$ as shown in the figure, and the moment with respect to Z is zero. These moments substituted in Euler's equations give (writing the $\dot{\omega}$ convention for angular acceleration and making $B=A$),

$$\left.\begin{aligned} A\dot{\omega}_x-(A-C)\omega_y\omega_z &= Mgh \sin \theta \cos \phi, \\ A\dot{\omega}_y+(A-C)\omega_x\omega_z &= -Mgh \sin \theta \sin \phi, \\ C\dot{\omega}_z &= 0. \end{aligned}\right\} \quad . \quad . \quad (77)$$

In these equations C is the Σmr^2 of the wheel, but A includes the Σmr^2 of both wheel and frame.

The necessary differential equations in $\dot{\theta}$, $\dot{\psi}$ and $\dot{\phi}$ for the solution of the problem are derived from Eqs. (76) and (77) by combination and integration, in such manner as to eliminate the principal angular velocities. However, the same results may be obtained more directly by applying general principles already deduced.

First, by the Conservation of Energy. The work done by the weight between the initial and any other value of θ is converted into kinetic energy of rotation about X and Y, there being no torque to change the rate of rotation about Z; that is, ω_z is constant and $\frac{1}{2}C\omega_z^2$ remains unchanged.

We therefore have

$$\tfrac{1}{2}A\,\omega_x^2+\tfrac{1}{2}A\,\omega_y^2=Mgh(\cos \theta_0-\cos \theta), \quad . \quad . \quad . \quad (78)$$

and eliminating the principal angular velocities by (76),

$$A\dot{\theta}^2+A\dot{\psi}^2 \sin^2 \theta=2Mgh\,(\cos \theta_0-\cos \theta). \quad . \quad (79)$$

This equation might be written directly from the fact that the work of the weight may be considered as converted into kinetic energy about any two rectangular principal axes in the plane XY. ON and the intersection of XY by the azimuth plane $Z'OZ$ constitute such a pair of axes. The kinetic energy about ON is $\frac{1}{2}A\,\dot{\theta}^2$, and that about the intersection of XY by the azimuth plane is $\frac{1}{2}A(\dot{\psi}\sin\theta)^2$.

Eq. (79) may also be obtained by multiplying the first of (77) by ω_x and the second by ω_y, and then adding the integrating between the limits θ_0 and θ.

Second, by the Conservation of Moments. From Eq. (55) we see that the principle of the conservation of moments applies to any axis for which the torque is zero. Under the conditions of the problem the initial moment of the momentum about the vertical axis Z' is $C\omega_z\cos\theta_0$ and there can be no change in the value of this moment because the weight has no torque with respect to Z'. For any other value of θ the moment of the momentum with respect to Z' consists of two parts, $C\omega_z\cos\theta$ and $(A\dot{\psi}\sin\theta)\sin\theta$, $A\dot{\psi}\sin\theta$ being the moment with respect to the intersection of the XY plane by the azimuth plane through Z and Z'. The sum of these two parts must be equal to the initial value, $C\omega_z\cos\theta_0$, and from this relation we get *

$$A\dot{\psi}\sin^2\theta = C\omega_z(\cos\theta_0-\cos\theta). \quad . \quad . \quad . \quad . \quad (80)$$

Third. From the third of (76) we have

$$\dot{\phi} = \omega_z - \dot{\psi}\cos\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

---

* To obtain this equation from (77) multiply the first by $\sin\phi$ and the second by $\cos\phi$, add, eliminate the angular velocities and accelerations, and integrate. The addition gives

$$A(\dot{\omega}_x\sin\phi+\dot{\omega}_y\cos\phi)+\omega_z(C-A)(\omega_y\sin\phi-\omega_x\cos\phi)=0, \quad . \quad . \quad (1)$$

and for the elimination we have from (76),

$$\omega_y\sin\phi-\omega_x\cos\phi=-\dot{\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\omega_x\sin\phi+\omega_y\cos\dot{\phi}=\dot{\psi}\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By differentiating (3) and dividing by dt we have

$$\begin{aligned}\dot{\omega}_x\sin\phi+\dot{\omega}_y\cos\phi &= \dot{\theta}\dot{\psi}\cos\theta+\ddot{\psi}\sin\theta-(\omega_x\cos\phi-\omega_y\sin\phi)\dot{\phi}\\ &= \dot{\theta}\dot{\psi}\cos\theta+\ddot{\psi}\sin\theta-(\omega_z-\dot{\psi}\cos\theta)\dot{\theta}\\ &= 2\dot{\theta}\dot{\psi}\cos\theta+\ddot{\psi}\sin\theta-\omega_z\dot{\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)\end{aligned}$$

Eqs. (2) and (4) reduce (1) to

$$A(2\dot{\theta}\dot{\psi}\cos\theta+\ddot{\psi}\sin\theta)=C\omega_z\dot{\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

whose integration (after multiplying by $\sin\theta$) between the limits θ_0 and θ gives (80).

in which ω_z is constant either by the principle of conservation of moments or by integration of the third of Eq. (77).

We thus have for the solution of the problem the three differential Eqs. (79), (80), (81). By eliminating $\dot{\psi}$ from the first two we may obtain an equation of the form

$$dt=f(\theta)d\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

from which θ is obtainable in terms of t. This value of θ in (80) gives

$$d\psi=f(t)dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

whose integration gives ψ in terms of t.

The problem may then be considered as completely solved, since the angular motion about Z is given by $\omega_z t$. If desired, however, the angle ϕ may be obtained from (81) after reduction to the form

$$d\phi=f(t)dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

The integration of these equations involves the use of the method by elliptic functions, but the general character of the motion may be shown from the differential equations without finding the time integrals.

38. Limits of the Nutation. The limits of θ which define the extent of the nutation, or oscillations in the azimuth plane, are obtained by making $\dot{\theta}=0$ in the equation $\dot{\theta}=f(\theta)$, derived from (79) and (80) by eliminating $\dot{\psi}$.

The solution for $\dot{\theta}$ gives

$$\left.\dot{\theta}=\pm\frac{\sqrt{(\cos\theta_0-\cos\theta)[2MghA\sin^2\theta-C^2\omega_z^2(\cos\theta_0-\cos\theta)]}}{A\sin\theta}\right\}, \quad (85)$$

and the zero value of $\dot{\theta}$ is given by either of the conditions

$$\cos\theta_0-\cos\theta=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

$$2MghA\sin^2\theta-C^2\omega_z^2(\cos\theta_0-\cos\theta)=0. \quad . \quad . \quad . \quad (87)$$

One of the limits is, therefore, the initial value of θ, and the other is obtained by solving (87) for θ. One of the roots of this equation gives $\cos\theta>1$, and, therefore, gives no value of θ. The other root gives a maximum value of θ and fixes the lower limit of the dip.

Indicating this maximum value of θ by θ' we have from (87),

$$\sin^2\theta'=\frac{C^2\omega_z^2(\cos\theta_0-\cos\theta')}{2MghA}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

For any assumed value of θ' the difference of the cosines varies inversely as ω_z^2. That is, the vertical dip necessary to reach θ' varies inversely as ω_z^2. The vertical dip from any particular value of θ_0 changes inversely with ω_z^2 and directly with $\sin^2\theta'$, but the dip is still a decreasing function of ω_z, as may be shown by the sign of $\frac{d\omega_z}{d\theta}$.*

39. The Precessional Velocity. From (80) we have

$$\dot{\psi}=\frac{C\omega_z(\cos\theta_0-\cos\theta)}{A\sin^2\theta}. \qquad (89)$$

Therefore the precessional velocity has the same sign as ω_z, being direct when ω_z is positive and retrograde when ω_z is negative.

For its upper limit $\theta=\theta_0$, giving

$$\dot{\psi}=0. \qquad (90)$$

For its lower limit $\theta=\theta'$ and $\dot{\theta}=0$. Then (79) gives

$$\cos\theta_0-\cos\theta'=\frac{A\dot{\psi}^2\sin^2\theta'}{2Mgh}, \qquad (91)$$

and this in (89) gives

$$\dot{\psi}=\frac{2Mgh}{C\omega_z}. \qquad (92)$$

Thus the maximum precessional velocity varies inversely as ω_z.

40. General Character of the Motion. By following both the nutation and precession we may see that a point on the axis of the gyroscope describes a curve on the surface of a sphere, between two horizontal circles at the limits of nutation, the curve being normal to the upper circle and tangent to the lower; somewhat like a cycloidal curve described by a point of a small circle of the sphere tangent to both the horizontal circles and rolling on the upper one.

---

* Write (88) in the form $\sin^2\theta=ax^2(\cos\theta_0-\cos\theta)$. Then

$$\frac{d\theta}{d(x^2)}=\left(\frac{a}{\sin\theta}\right)\frac{(\cos\theta_0-\cos\theta)}{(2\cos\theta-ax^2)}=\left(\frac{a}{\sin\theta}\right)\frac{(\cos\theta_0-\cos\theta)^2}{[2\cos\theta\,(\cos\theta_0-\cos\theta)-\sin^2\theta]}.$$

The numerator is always positive, and the denominator

$$\begin{aligned}&=2\cos\theta_0\cos\theta-2\cos^2\theta-\sin^2\theta\\&=(-\cos^2\theta_0+2\cos\theta_0\cos\theta-\cos^2\theta)-\cos^2\theta-\sin^2\theta+\cos^2\theta_0\\&=\cos^2\theta_0-1-(\cos\theta_0-\cos\theta)^2,\end{aligned}$$

which is always negative.

2. After spinning the wheel give the gyroscope an initial angular velocity whose components are $\dot{\theta}_0$ and $\dot{\psi}_0$.

By the conservation of energy we have, Eq. (79),

$$A\dot{\theta}^2+A\dot{\psi}^2\sin^2\theta=A\dot{\theta}_0{}^2+A\dot{\psi}_0{}^2\sin^2\theta_0+2Mgh\,(\cos\theta_0-\cos\theta), \quad . \quad (93)$$

and by the conservation of moments with respect of Z', Eq. (80),

$$A\dot{\psi}\sin^2\theta=A\dot{\psi}_0\sin^2\theta_0+C\omega_z(\cos\theta_0-\cos\theta). \quad . \quad . \quad (94)$$

If we write $2Mgha\cos\theta_0$ for the constant part of (93) and $C\omega_z\,b\cos\theta_0$ for the constant part of (94) these equations may be simplified in form as follows:

$$A\dot{\theta}^2+A\dot{\psi}^2\sin^2\theta=2Mgh(a\cos\theta_0-\cos\theta), \quad . \quad . \quad (95)$$

$$A\dot{\psi}\sin^2\theta=C\omega_z(b\cos\theta_0-\cos\theta). \quad . \quad . \quad . \quad (96)$$

41. Limits of Nutation. Making $\dot{\theta}=0$ and eliminating $\dot{\psi}$ we have

$$2Mgh(a\cos\theta_0-\cos\theta)\,A\sin^2\theta-C^2\omega_z{}^2(b\cos\theta_0-\cos\theta)^2=0, \quad . \quad (97)$$

whose roots give the limiting values of θ.

Substituting $1-\cos^2\theta$ for $\sin^2\theta$ we see that this is a cubic equation. In general its three roots give $\theta'>\theta_0$, $\theta''<\theta_0$, and $\cos\theta>1$; θ' designating the lower limit and θ'' the upper.

41. The Precessional Velocity of the Limits of Nutation. From (96) we have for $\dot{\psi}'$ and $\dot{\psi}''$, corresponding to θ' and θ'',

$$\dot{\psi}'=\frac{C\omega_z(b\cos\theta_0-\cos\theta')}{A\sin^2\theta'}, \quad . \quad . \quad . \quad . \quad . \quad (98)$$

$$\dot{\psi}''=\frac{C\omega_z(b\cos\theta_0-\cos\theta'')}{A\sin^2\theta}. \quad . \quad . \quad . \quad . \quad . \quad (99)$$

There are four cases. (*a*) The initial angular velocity may be consistent with the motion already described in 1, $\dot{\psi}$ and $\dot{\theta}$ vanishing together at the upper limit of the nutation. This requires $b\cos\theta=\cos\theta''$ in (99).

(*b*) When the initial velocity is wholly in nutation—that is, $\dot{\psi}_0=0$—we have $b=1$ in (96), or, rather, (94) and (96) take the form of (80), and then $\dot{\psi}'$ and $\dot{\psi}''$ have opposite signs in (98) and (99). This reversal of precession causes a point on the axis of the gyroscope to describe a series of equal loops on the surface of a sphere, the path of the point lying between two horizontal circles and tangent to both at the linits of nutation. This path is somewhat like a looped epitrochoid bent into the spherical surface.

(*c*) When $b \cos \theta_0 > \cos \theta''$ the angular velocities $\dot{\psi}'$ and $\dot{\psi}''$ have the same sign. The path of a point on the axis is then tangent to both the limiting circles, with continuous precession in one direction and a point of inflection between any two consecutive tangencies. The comparison with an epitrochoid still exists, but in this case the epitrochoid is without loops.

(*d*) The gyroscope may spin with uniform precession and θ constant, and then we have from (95) and (96),

$$\dot{\psi} = \dot{\psi}_0 = \frac{2Mgh(a-1)}{[C\omega_z(b-1)]}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

But a and b are functions of $\dot{\psi}_0$; to determine when this motion is possible we may deduce a condition for $\dot{\psi}_0$ as follows:

For the condition $\dot{\theta} = 0$ we have from (76) by differentiation,

$$\dot{\omega}_x = \dot{\psi} \sin \theta \cos \phi \dot{\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

$$\dot{\omega}_y = -\dot{\psi} \sin \theta \sin \phi \dot{\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

These equations enable us to eliminate the principal angular accelerations and velocities from Euler's equations applied to the gyroscope (77). Then by $\dot{\phi} = \omega_z - \dot{\psi} \cos \theta$ (3d Eq. (76)), we finally get for the first two of these equations under the given conditions,

$$\left.\begin{aligned} \sin \theta\, (A\dot{\psi}^2 \cos \theta - C\omega_z\dot{\psi} + Mgh) \cos \phi = 0, \\ \sin \theta\, (A\dot{\psi}^2 \cos \theta - C\omega_z\dot{\psi} + Mgh) \sin \phi = 0. \end{aligned}\right\} \quad . \quad . \quad (103)$$

Since these equations must be satisfied for all values of ϕ we have

$$\sin \theta\, (A\dot{\psi}^2 \cos \theta - C\omega_z\dot{\psi} + Mgh) = 0, \quad . \quad . \quad . \quad (104)$$

as the necessary condition for uniform motion of the gyroscope. This condition is satisfied

First. When $\sin \theta = 0$. That is, Z is vertical, with the center of gravity either above or below the fixed point.

Second. When $\cos \theta = 0$ Eq. (104) is satisfied by $C\omega_z\dot{\psi} = Mgh$, and the axis spins in a horizontal plane with

$$\dot{\psi} = \frac{Mgh}{C\omega_z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

which is a mean of the limiting precessional velocities when the axis falls from rest (90) and (92).

Third. In the general case the equation

$$A\dot{\psi}^2\cos\theta - C\omega_z\dot{\psi} + Mgh = 0, \quad . \ . \ . \ . \quad (106)$$

gives two values of $\dot{\psi}$, which are equal, unequal, or imaginary, according as

$$C^2\omega_z^2 =,\ >,\ \text{or}\ < 4A\cos\theta Mgh. \quad . \ . \ . \quad (107)$$

Fourth. If $\omega_z = 0$ we have the case of the conical pendulum, with $\alpha = \beta$; and from (106) we have for the precessional velocity,

$$\dot{\psi} = \sqrt{-\frac{Mgh}{A\cos\theta}}. \quad . \ . \ . \ . \ . \ . \ . \quad (108)$$

The period of revolution is therefore

$$T = 2\pi\sqrt{-\frac{A\cos\theta}{Mgh}}. \quad . \ . \ . \ . \ . \ . \ . \quad (109)$$

For a small displacement of the conical pendulum from the vertical $\alpha = \beta = 180° - \theta$, we may make the approximation $\cos\theta - 1$, and then

$$\dot{\psi} = \sqrt{\frac{Mgh}{A}}. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (110)$$

The time of the revolution under this approximation is therefore

$$T = 2\pi\sqrt{\frac{A}{Mgh}} = 2\pi\sqrt{\frac{(k^2+h^2)}{gh}}, \quad . \ . \ . \ . \quad (111)$$

which agrees with the period found for the compound pendulum under the assumption of isochronal vibrations through a small arc.

The value of $\dot{\psi}$ in (108) approaches ∞ as θ approaches 90°, and is imaginary when $\theta < 90°$.

EXERCISES TO BE PERFORMED BY THE STUDENT IN THE LABORATORY

Note.—The introduction of Laboratory Exercises in the Course of Natural and Experimental Philosophy at the Military Academy seemed advisable for the purpose of crystallizing in the student's mind the fundamental principles of the course in a wholly practical manner.

It is not intended that this innovation replace the theoretical treatment of the subject, but rather, that it go hand in hand with the theory.

At many of the Technical Schools the course in laboratory work of this nature is combined with "Precision Measurements." The latest tendency, however, seems to be away from this idea and towards a separation of the two, so that the student in beginning will have his attention on mechanical principles and not on a certain numerical result. Later, precision measurements are developed in the more advanced laboratory work.

These experiments are prepared in accordance with this latest tendency in laboratory work and for our purpose are intended solely for the development of principles and are not "precision experiments."

Experiments 1 to 13 inclusive are similar to those devised by Professor Duff of Worcester Polytechnic Institute. The exercises outlined here have been rewritten and somewhat amplified and improvements in the apparatus itself have been incorporated, which render the results entirely satisfactory for the intended purpose. *Forms* for recording data have been arranged so as to systematize the work both for the benefit of the student and the instructor.

The apparatus shown in the half-tones is not difficult to make and can be devised with a few additions to the stock usually on hand. The sets furnished the U. S. Military were made by Mr. C. D. Haigis, now with the Department of Physics at Worcester Polytechnic Institute. These sets are extremely well made, and Mr. Haigis has incorporated many improvements which bring results of surprising accuracy for apparatus of such extreme simplicity.

Experiments 14 to 16 with the subheads are original simple experiments in *optics* and *sound*, based upon work done at Cornell

University. Professor Richtmyer was kind enough to offer suggestions which are incorporated therein.

While the simple principles are covered by the experiments outlined here it would appear to be advantageous to amplify the course when time permits. A sufficient number of experiments are here given to serve as a guide for additional laboratory problems designed to meet any special set of conditions.

An examination of the method employed in these exercises will show the underlying purpose is to make the student think for himself while he is, at the same time, doing practical work which should fix the simple fundamental principles in his mind definitely and permanently.

HALSEY DUNWOODY,
Acting Professor of Philosophy, U. S. M. A.

July, 1917.

Exercise 1. The Vernier.

Principle of Vernier. In general

$$nv=(n\pm 1)s,$$

where

n = No. of vernier divisions,
v = length of one vernier division and
s = length of one scale division. Hence,

$$v-s=\pm\frac{1}{n}s.$$

Make Direct Vernier. To read $\frac{1}{10}$ of $\frac{1}{8}$ in.

Mark off on cardboard strip (vernier) a length $=\frac{9}{8}$ in.

Divide this length into 10 equal parts, using cross-section paper for this purpose.

Draw divisions in sharp, hard lead pencil lines, being careful that lines are perpendicular to edge of cardboard scale.

Number the divisions 0, 5, 10 in the direction of main scale (ruler.)

Make Retrograde Vernier. Divide $\frac{11}{8}$ inches into 10 equal parts.

How must this vernier scale be numbered to make the vernier a direct-reading instrument?

Fasten the strips on the sliding block.

Test by running vernier along main scale to determine whether vernier is of proper length.

Measurements. 1. Place scale and vernier in contact in an upright position on a smooth surface and take the reading of the zero point of the vernier. (Zero reading.)

2. Place the specimen to be measured under the lower end of the vernier block and take the reading.

3. What is the dimension of the specimen thus measured?

Repeat the same measurements with retrograde vernier.

4. Record observations on data sheet for Exercise 1.

5. Put name on vernier and hand in with data sheet.

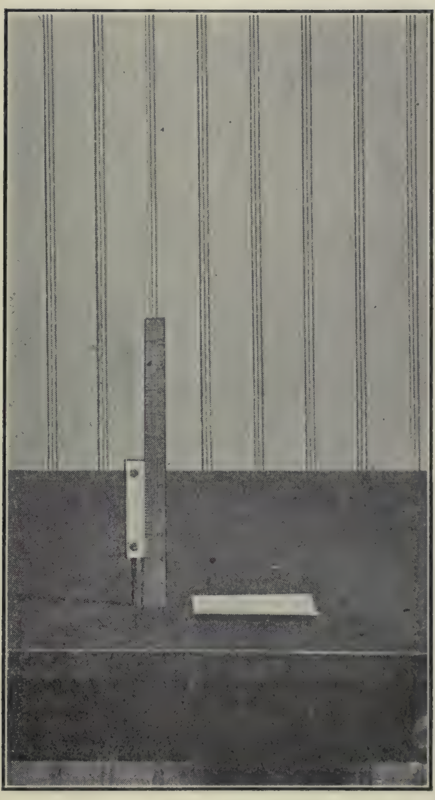

FIG. 1.

Name .... Section .... Exercise ....

Date..................

U. S. M. A. LABORATORY

(Department of Natural and Experimental Philosophy)

Subject of Experiment

DIAMETER			LENGTH		
Reading for Zero	*Reading for Length*	*True Length*	*Reading for Zero*	*Reading for Length*	*True Length*
Direct Vernier Average			Direct Vernier Average		
Reading for Zero	*Reading for Length*	*True Length*	*Reading for Zero*	*Reading for Length*	*True Length*
Retrograde Vernier Average			Retrograde Vernier Average		

Instrument	*Smallest Scale Division*	*Number of Vernier Divisions*	*"Least Count" of Vernier*

Form 1

NOTES

Exercise 2. The Path of a Projectile. (Neglecting Air Resistance.)

General Statement. (See Fig. 2.) Since the vertical distance passed over by the projectile (disregarding air resistance) is equal to that of a falling weight, $y=\frac{1}{2}gt^2$ and the horizontal velocity is constant (the only force acting being vertical, gravity); $x=v_1t$. We have, eliminating t, $\frac{x^2}{y}=\frac{2v_1^2}{g}$ from which v_1 is obtained for any x and y.

Adjustments. Lines of board must be vertical and horizontal. (Plumb line.)

Groove, horizontal.

Spring.

Pocket.

Measurements. 1. Observe the x coordinate of the projectile when it crosses every other horizontal line. Have line of sight perpendicular to board. Record, as per data sheet:

1. Actual readings on odd-numbered lines.
2. Corrected distances, $y-y_0$ and $x-x_0$, referred to x_0, y_0, the coordinates of the point where the ball leaves the groove.
3. Reduce $y-y_0$ and $x-x_0$ to centimeters.

Calculations. (See data sheet.)

Mean value of $\frac{x^2}{y}$.

Calculate v_1.

Calculate t at line XIII ($x=v_1t$, x and v_1 are known).

Calculate v_2, the vertical velocity at line XIII ($v_2=gt$).

Calculate v, the resultant velocity at line XIII ($v=\sqrt{v_x^2+v_y^2}$).

Find tangent θ at line XIII from velocities. (θ is slope of trajectory.)

Find tangent θ from a curve of the path plotted from data obtained in experiment.

What kind of a curve is it?

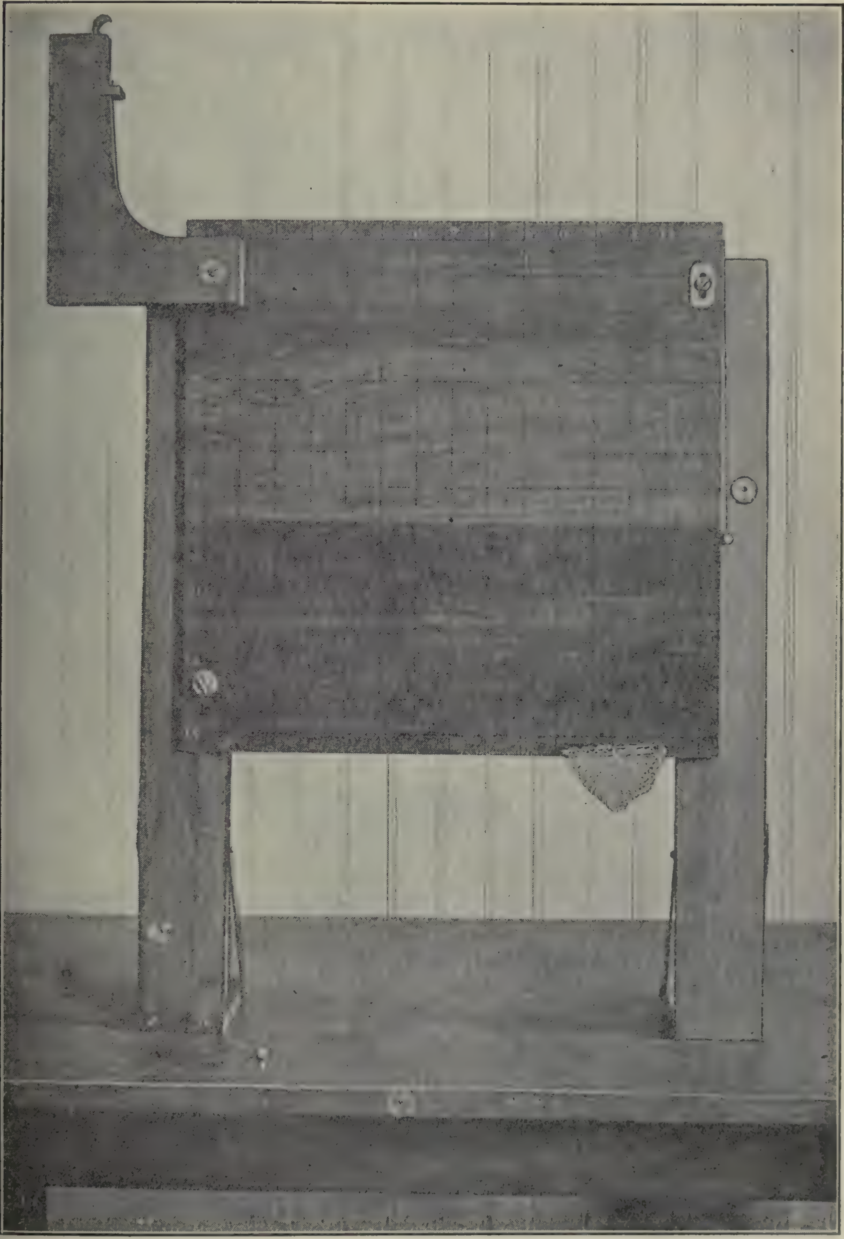

FIG. 2.

Name . . . . Section . . . . Exercise . . . .

Date. . . . .

U. S. M. A. LABORATORY

(DEPARTMENT OF NATURAL AND EXPERIMENTAL PHILOSOPHY)

Subject of Experiment

BOARD UNITS									Average Reduced to Centimeters		
Trials	*Line III*	*Line V*	*Line VII*	*Line IX*	*Line XI*	*Line XIII*	*Line XV*		$x-x_0$ *cm*	$y-y_0$ *cm*	$\frac{x^2}{y}$
1											
2											
3											
4											
5											
Average											

$x_0=$ $y_0=$ Average =

Initial velocity, $v_1=$

Time at line XIII $t=$

Vertical velocity at line XIII $=v_2=$

Total velocity at line XIII, $v=$

Tangent θ (from velocities) =

Tangent θ from curve =

Nature of curve?

Form 2.

NOTES

Exercise 3. Centripetal Force

General Statement. (See Figs. 3 and 3*a*.) Since a body moving in a circle, with constant speed v is subject to a constant acceleration $a=\frac{v^2}{r}$ directed toward the center a constant force $F=m\frac{v^2}{r}$ is required to keep the body moving in the circle. This force is exerted in this case by a stretched spring. F is to be calculated from m, v, and r, and to be compared with the value of F corresponding to the stretch of the spring obtained from a calibration curve for the spring, as in Fig. 3*a*.

Adjustments. Needle points sharp.

Adjust the index wire so that it is opposite the pin projecting from the lead block when the latter hangs freely.

Make the axis vertical (no tendency of weight to oscillate).

Spring must be approximately horizontal during the motion.

Measurements. (See data sheet.) Measure r from center of axis to center of weight by mirror scale.

Length of spring from end of hook to end of hook. (Over all.)

Get number of rotations in three minutes three times.

Calculations:

1. Calculate the acceleration a.
2. Calculate the force F.
3. In what force unit is it expressed?
4. Calibration of spring. (Fig. 3*a*)
    (*a*) Measure lengths with three different weights attached. Avoid parallax.
    (*b*) Draw the calibration curve of the spring.

Precautions. Avoid parallax in measurements and in the observation of the passages over the index wire.

Give impulses midway between the passages.

Avoid jerks at start and in stopping.

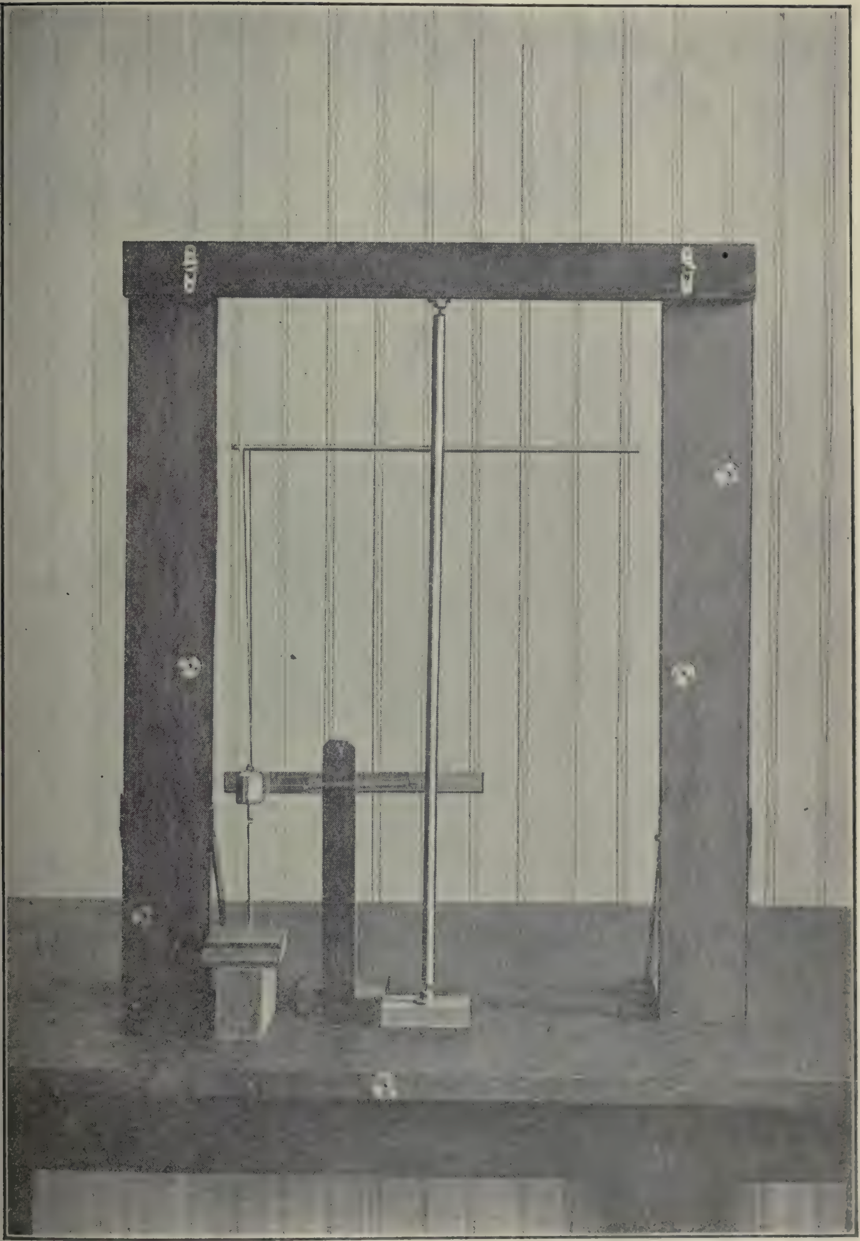

FIG. 3.

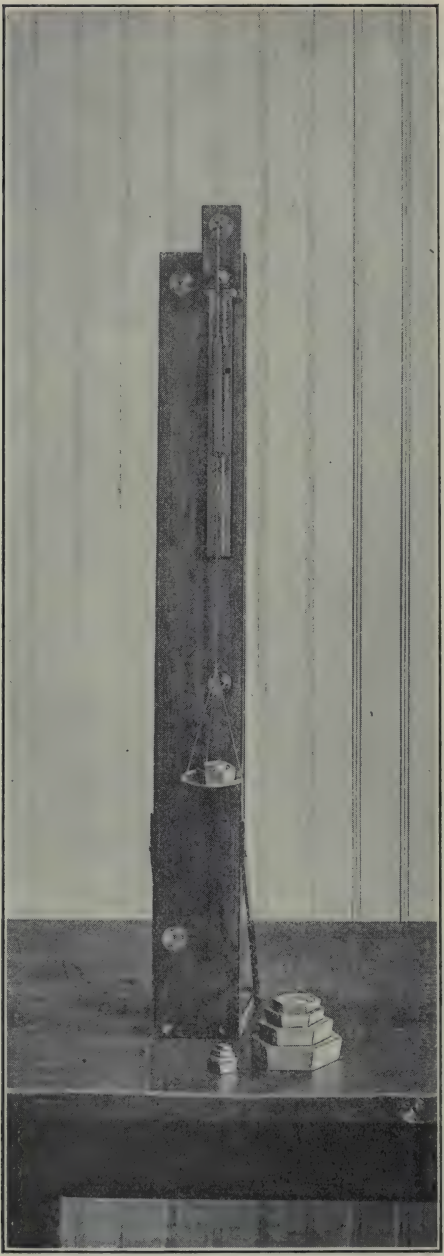

FIG. 3*a*

Name................ Section................ Exercise................

Date..................

U. S. M. A. LABORATORY

(DEPARTMENT OF NATURAL AND EXPERIMENTAL PHILOSOPHY)

Subject of Experiment.

No. of Rev. in 3 min.
Average

No. per sec. =

Measurement of *r*

Axis		*Lead*	
Left Side	*Right Side*	*Left Side*	*Right Side*

$r =$

Length of Spring

Average Length	

Calibration of Spring.

Spring No...............

Mass of Lead =

Speed (calculated) =

Acceleration (calculated) =

Force (calculated) =

Force (from calibration) =

% Difference =

Form 3.

NOTES

Exercise 4. Composition of Forces; to Find the Resultant

General Statement. (See Fig. 4.) In this experiment three forces represented by weights, all three acting at a point, are held in equilibrium by a fourth force produced by a stretched spring. Therefore the resultant of the first three forces should be equal in magnitude to the force exerted by the spring, and in the opposite direction.

Adjustments. Level board with plumb line.

Let no two cords be in the same line.

Have the springs stretched from one and one-half to three times. No more.

Measurements. Get direction of forces (the angle a measured counter-clockwise from the horizontal), by projecting lines of cords on board.

Measure length of the spring (over all). Tabulate forces and their angles with the horizontal.

Copy the figure on cross-section paper.

Calibrate spring and draw the calibration curve on cross-section paper.

Calculations. Find the resultant of the first three forces.

1. Graphically by constructing a force polygon.
2. Analytically by calculation from the forces and their angles with the horizontal, resolving the forces into their x and y components.
3. Experimentally from the length and calibration of the spring.

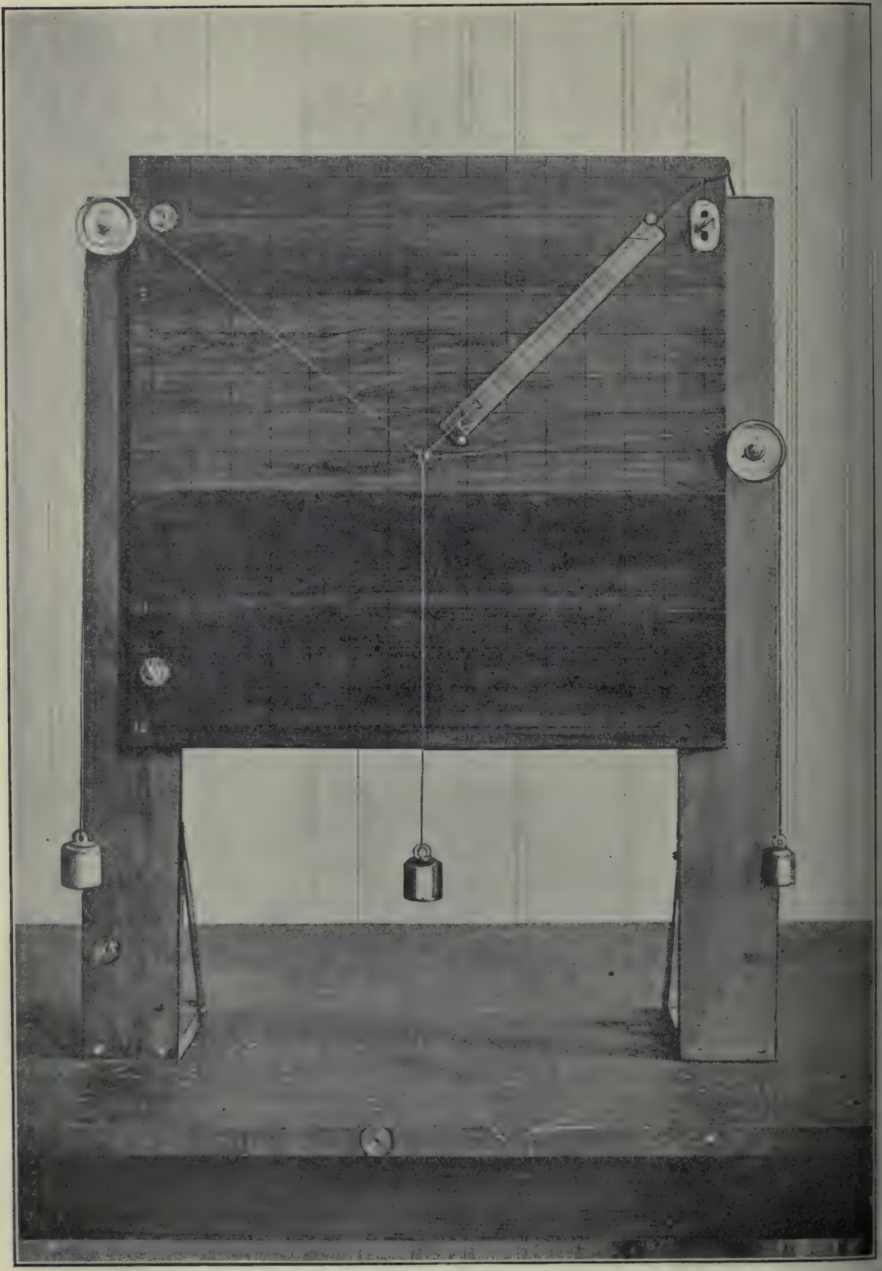

FIG. 4.

Name.................. Section.................. Exercise..................

Date....................

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(Department of Natural and Experimental Philosophy)

Subject of Experiment.

Forces	α	$F \cos \alpha$	$F \sin \alpha$
Spring	Length = Inclination =	$x =$	$y =$

Calibration of Spring

Resultant

Method	R	θ
Analytical		
Graphical		
Experimental		

$x = \Sigma F \cos \alpha$

$y = \Sigma F \sin \alpha$

$R^2 = x^2 + y^2$

Form 4.

NOTES

Exercise 5. Impact of Spheres

General Statement. (See Fig. 5.) At impact there is no change of the total momentum.

$$m_1v_1+m_2v_2=m_1v'_1+m_2v'_2.$$

Where m_1v_1 and m_2v_2 are the momenta of the spheres before impact and $m_1v'_1$ and $m_2v'_2$ are the momenta after impact.

The apparatus consists of two pendulum-carrying spheres.

The velocity of the sphere in its path $v_1=\sqrt{2gh_1}$, where h_1 is the vertical drop and similarly for the other velocities.

For the calculation of h we have l, the length of the pendulum, x the horizontal distance of swing, giving

$$l^2=(l-h)^2+x^2$$

$$0=-2lh+h^2+x^2.$$

In this experiment the swing is short and therefore h is relatively small as compared with l and x. Thus we may disregard h^2, which gives $h=\frac{x^2}{2l}$ as an approximation.

NOTE. This approximation for h is used in the deduction of the equation for the location of images formed by mirrors and lenses.

Adjustments. Centers of spheres must be at the same level.

Planes of the V's formed by the supporting threads must be parallel and both must be perpendicular to the board.

Branches of the V's must be equal.

Measurements. Get length of each pendulum from support to the center of the ball.

Horizontal distance through which m_1 swings before impact.

Horizontal distance through which m_1 swings after impact.

Horizontal distance through which m_2 swings after impact.

Calculations:

1. Velocities v_1, v'_1, and v'_2.
2. Momentum of m_1 before impact (that of m_2 is zero).
3. Sum of momenta of m_1 and m_2 after impact.
4. Kinetic energy of m_1 before impact. ($\frac{1}{2}mv^2$.)
5. Sum of kinetic energies of both after impact.
6. Find the ratio $(v'_2-v'_1)\div v_1$.

What is this ratio called?

7. Should there be any decrease in total momentum? Why?
8. Should there be any decrease in total energy? Why?

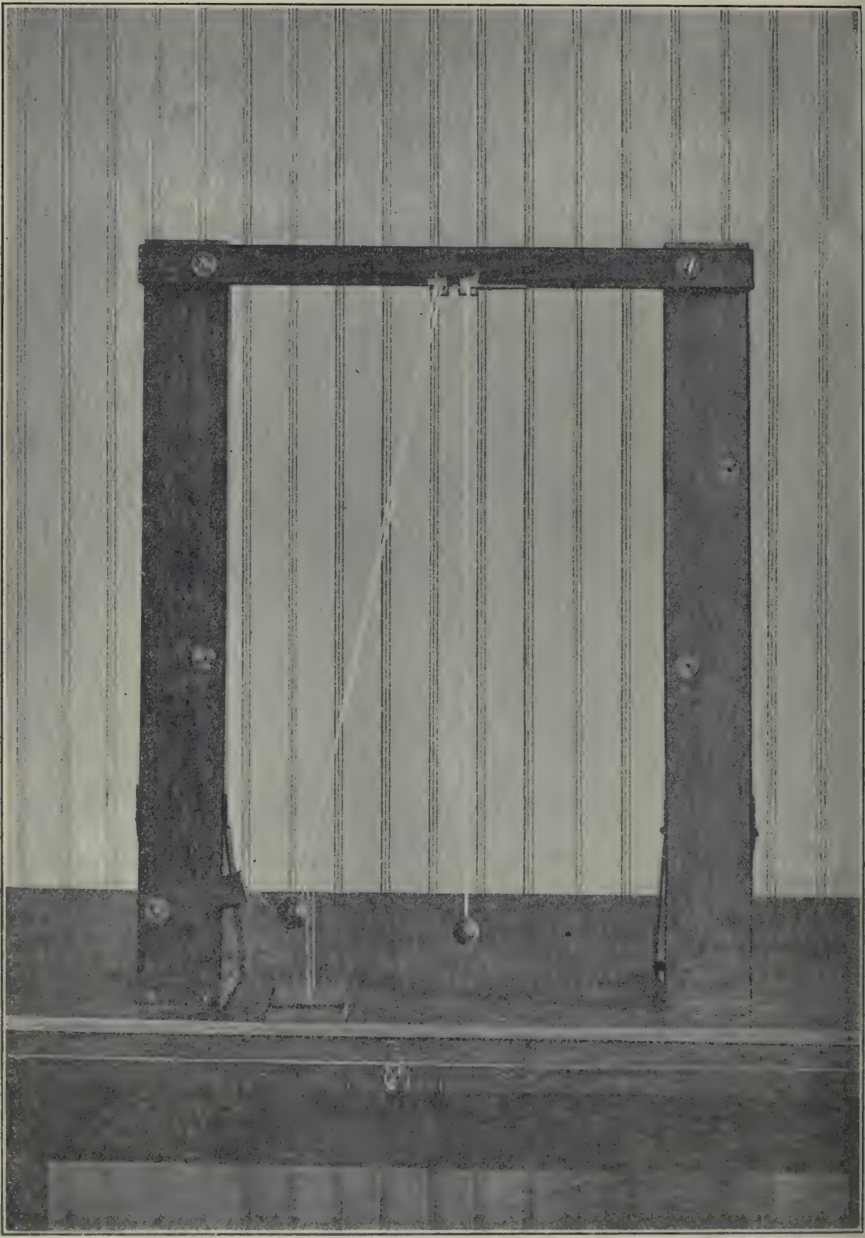

Fig. 5.

Name .... Section .... Exercise ....

Date ....

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(Department of Natural and Experimental Philosophy)

Subject of Experiment

BEFORE IMPACT

	Reading a	*Reading b*	*Hor. Dist.*	*Vert. Dist.*	*Velocity*	*Mass*	*Momentum*	*K. E.*
1								
2								
						Total		

Length of Pend =

AFTER IMPACT

	Reading a	*Reading b*	*Hor. Dist.*	*Vert. Dist.*	*Velocity*	*Mass*	*Momentum*	*K. E.*
1								
2								
						Total		

$(v_2{}^1 - v_1{}^1) \div v$ =

Name of Ratio =

Ans. to question 7.....................

Ans. to question 8.....................

Form 5

NOTES

Exercise 6. Angular Acceleration and Moment of Force

General Statement. (See Fig. 6.) In order to verify the relation between moment of force, moment of inertia and angular acceleration, we obtain in this experiment by direct observation the time of descent t through the distance h of a mass m attached to a revolving disk carrying two lead cylinders. The value of t thus obtained is compared with the value of t calculated from the equations involving I, L and α referred to below.

From the dimensions of disk, blocks and axis the total moment of inertia can be found and from m and the radius of the axis r, the moment of force L is found. From this α can be found ($L = I\alpha$) where I is the total amount of inertia. And t is given by $\phi = \frac{1}{2}\alpha t^2$, ϕ being the total angle through which the wooden disk has turned. (Radians.)

Adjustments. Needle points sharp.

Disk horizontal (no vibration with one block on it).

Measurements. Get time of descent five times.

Radii of lead cylinders disk and axle.*

Distances of the lead cylinders from the axis.

Get h, the distance of descent of m.

Calculations:

1. Moment of inertia of the disk about its axis. $\frac{1}{2}Mr^2$.
2. Moment of inertia of each lead cylinder about its axis. $\frac{1}{2}Mr^2$.
3. Moment of inertia of each lead cylinder about the axis of the disk. $\left(\frac{Mr^2}{2} + Mh^2\right)$.
4. Moment of inertia of the axle about its axis.
5. Total amount of inertia.
6. Moment L, allowing for one-half the thickness of the thread.†
7. Angular acceleration.
8. Angular displacement, $\left(\frac{h}{r}\right)$, h = length of rectified arc.
9. Time of descent.
10. Final angular velocity $= \frac{1}{2}\omega t$.
11. Loss of potential energy of m, (mgh).
12. Gain of kinetic energy of disk and weights. $\frac{1}{2}\omega^2\Sigma mr^2$ or $\frac{1}{2}\omega^2 I$.

---

* r for axle = .555 cm.

† r for thread = .006 to .015, according to thread.

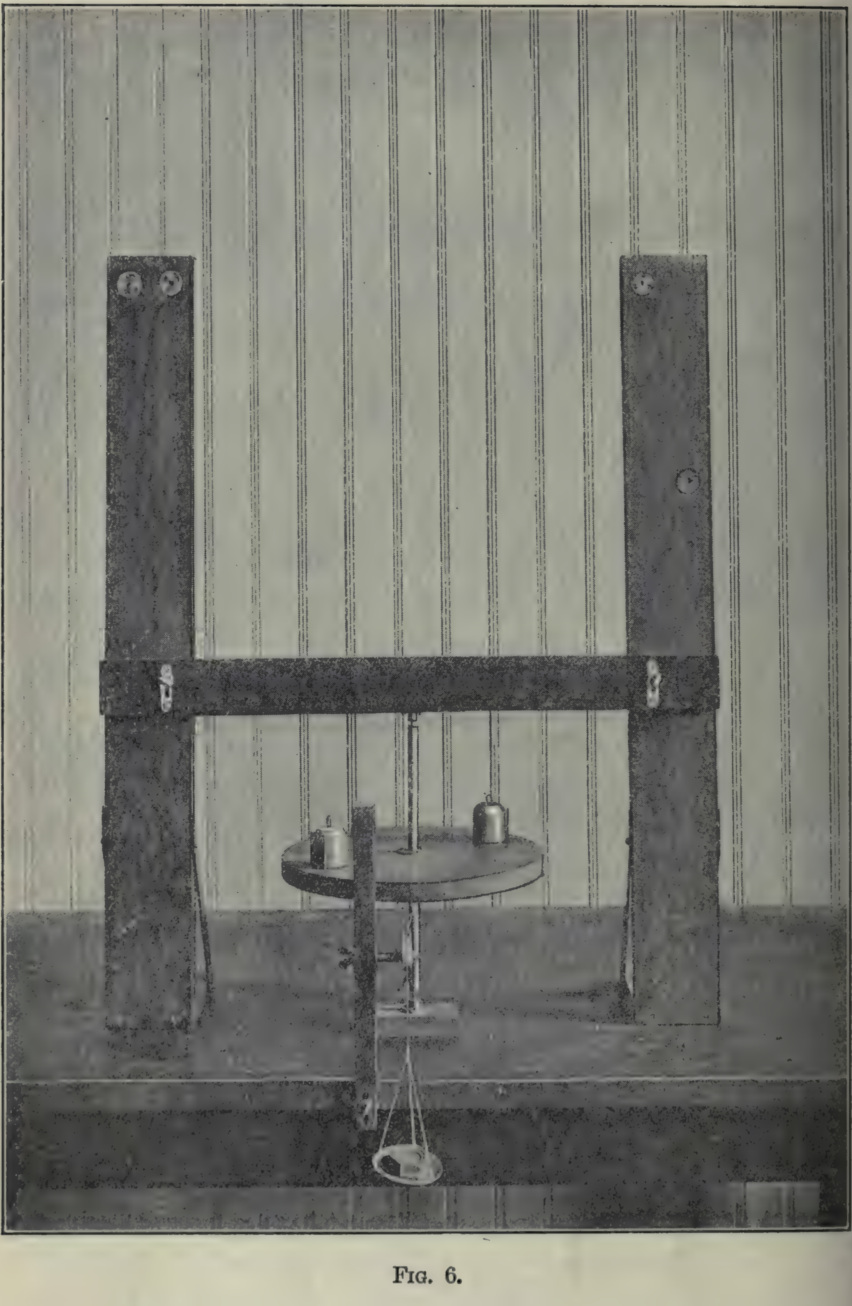

FIG. 6.

Name .... Section .... Exercise ....

Date ....

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Subject of Experiment

	Side near axis	*Side away from axis*	*Radius*	*Distance from axis*	*Mass*	*Moment of Inertia*
Lead cylinder						
Lead cylinder						
Disk						
Axle						
					Total	

Hanging Weight =

Distance of Descent =

Time of Descent

1........ 3........

2........ 4........

5........ Mean =

Thickness of Thread =

L = moment of force =

$\propto$ = angular acceleration =

θ = angular displacement =

t_1 time of descent (calculated) =

ω final angular velocity =

K. E. gained =

P. E. lost =

Form 6

NOTES

Exercise 7. Friction and Centripetal Force

General Statement. (See Fig. 7.) Coefficient of friction $\mu = \frac{F}{P}$, where F is the force parallel to the surface of contact, required to start the motion, and P the normal force between the two surfaces.

If friction just supplies the necessary force toward the center, then

$$\frac{mv^2}{r} = mr\omega^2 \quad (v = \omega r) \text{ or}$$

$$mr\omega^2 = \mu P. \qquad \text{See reference text.}$$

Adjustments. Same as in Experiment 6.

Axis must be vertical.

Have pegs to stop disk.

Measurements. Distance from center of friction specimen to the center of the axis.

Time of descent to the floor, T. (Mean of 5 trials.)

Distance of descent h.

Time of descent until slipping takes place (t).

Calculations:

1. Total angular displacement ϕ.
2. Angular acceleration, α.
3. Velocity (angular) at slip, ω_t.
4. Centripetal force, F.

Normal force, P.

Coefficient of friction, μ.

NOTE. Use same units in numerator and denominator.

In finding T the friction specimens should be in position on the disk, but the exact position of the small disk is unimportant.

5. Find the angle of slip θ of the same specimens on an inclined plane. Should there be any relation between μ and this angle?

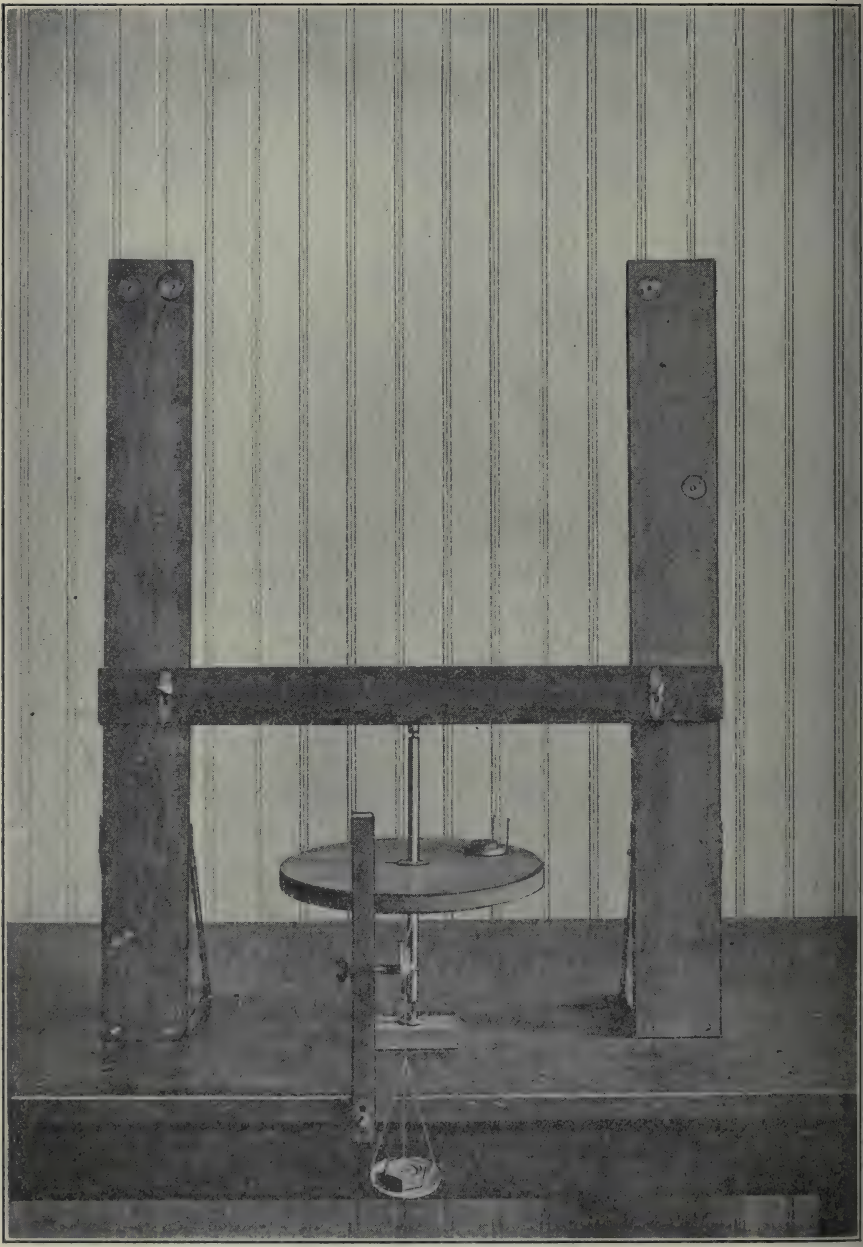

FIG.

Name .... Section .... Exercise ....

Date ....

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Subject of Experiment

	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Average
Time of descent to floor, T						
Time of descent to slip, t						

Distance from center of friction specimens to center of axle =

Distance of descent to floor =

Radius of axle $+\frac{1}{2}$ thickness of thread =

Total Angular Displacement =

Angular Acceleration =

Angular Velocity at Slip =

Centripetal Force =

Normal Force P =

Coefficient of Friction, μ =

Inclined Plane Method

Height at slip =

Base at slip =

θ =

Relation asked for in (6) =

Form 7

NOTES

Exercise 8. Parallel Forces

General Statement. (See Fig. 8.) 1. A very light frame free to rotate about a nail at its center is acted upon by four parallel forces, all in a vertical direction. Three forces are produced by hanging weights and the fourth is produced by a spring, pulling upward as shown in the figure. The resultant moment is zero (system in equilibrium) or $W_1l_1 \pm W_2l_2 \pm W_3l_3 \pm F_sl_4 = 0$.

(*a*) From measured values of W and l, F_s, the force exerted by the spring, may be computed and compared with the value, as obtained from the calibration curve of the spring.

2. In the second part of this exercise, the problem is to find first, by a graphical diagram to scale on cross-section paper, the position of the three action lines and intensity of the resultant reaction of the system of forces. Check this result by the analytical method.

Adjustments. Lines of the board must be vertical and horizontal.

Spring should be vertical and well stretched but not more than three times its length.

Measurements. Calibration of the spring.

Diagram on cross-section paper.

Measure the perpendicular distances from the action line of the forces to the nail (lever arms).

Weights.

Calculations:

1. Take forces two at a time around the quadrilateral, using the force of the spring last. Diagram on cross-section paper.

2. Take moments about the point (0 : 0) in vertical edge on board, and calculate the coordinate of the position of the center of moments by the relation in form $\bar{x} = \frac{Fl}{R}$, where R is the total reaction at the nail.

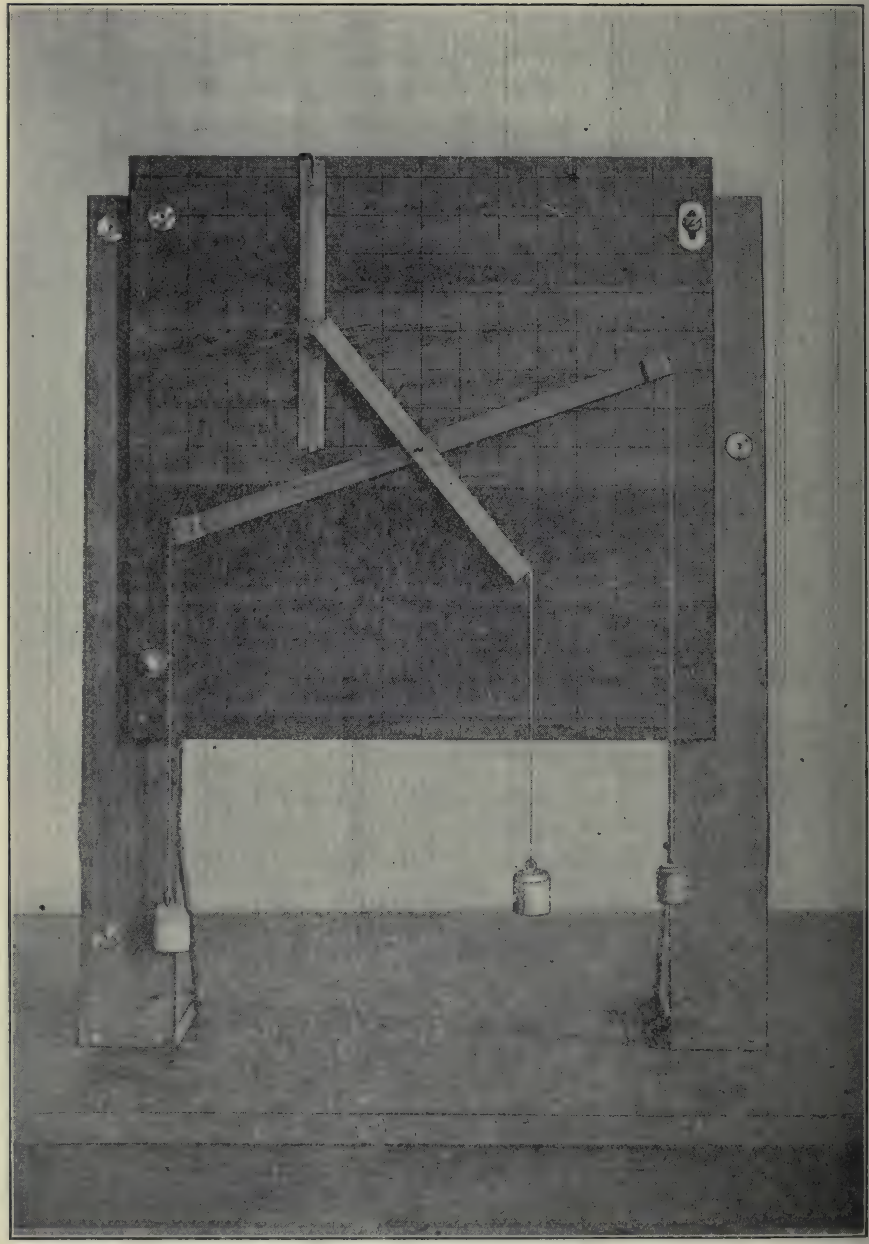

FIG. 8.

Name .... Section .... Exercise . ...

Date..... ....

U. S. M. A. LABORATORY

(Department of Natural and Experimental Philosophy)

Subject of Experiment

Calibration of spring

	Length	*Weight*
1		
2		
3		

Weights	Distance from nail	Moments
	Total	

Length of spring	Distance from nail
	$x =$

$$F_s = \frac{\Sigma(\text{moments})}{x} = \underline{\qquad\qquad} =$$

F_s from calibration =

Forces	x	Moments

$$\bar{x} = \frac{\Sigma Fl}{R} = \underline{\qquad\qquad} =$$

$R =$

x coordinate of nail =

Form 8

NOTES

Exercise 9. Equilibrium of a Body

General Statement. (See Fig. 9.) Body is acted on by five forces, namely:

Weight of the body at the center of gravity;
Suspended weight;
Cord and pulley;
Spring;
Reaction of the axis.

The purpose of this experiment is to verify the equations expressing the conditions for equilibrium.

Adjustments. Find the center of gravity of the body and mark it.

Make lines of the board horizontal and vertical.

Calibrate the spring used. (Cross-section paper.)

Attach forces in lines as far from the axis as possible.

Measurements. Get distances of forces from the axis by thread.

Measure angles by protractor.

Plot the system of forces and the position of the nail.

Calculations:

1. Using the necessary equations which state the conditions of equilibrium in this case, i.e., the summation of the horizontal and vertical components of the forces and the equation for the moments, solve for the force which the spring exerts, the reaction of the axis and the inclination of the reaction to the horizontal.

2. Check the force of the spring from the calibration.

3. Find experimentally the reaction and the angle at which it acts by removing the nail from the hole in the big board. Tie a thread to the nail which is now in the hole in the quadrilaterial only and which is free to move. Pass the other end of the thread over a pulley and tie to a scale pan.

Adjust the position of this thread by moving the pulley and the weights in the scale pan until the quadrilaterial remains in its original position with the nail in front of the hole in the big board.

Compare the weights in the scale pan with the value of R (reaction at nail, calculated in the first part of this experiment) also compare the angle made by the action of this force with the computed angular value.

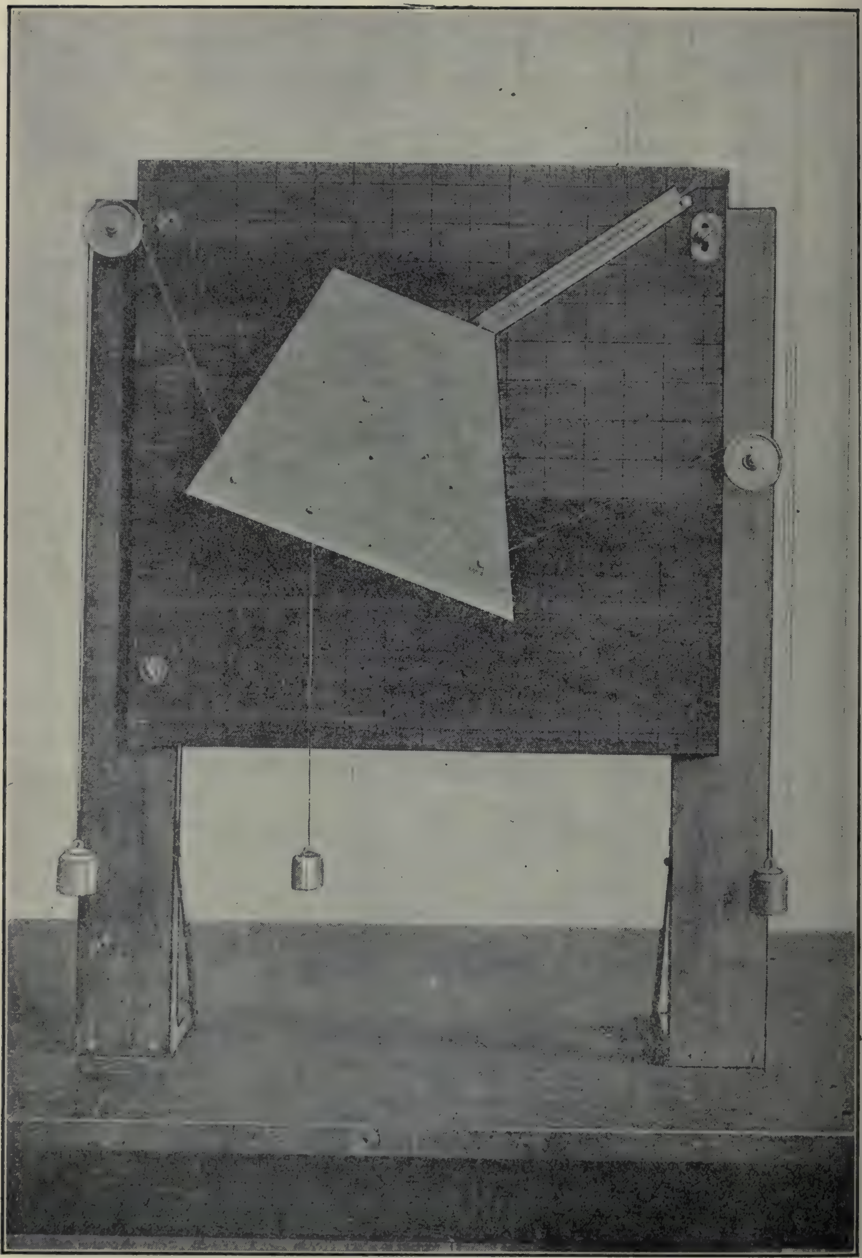

FIG. 9.

Name...................... Section.............. Exercise.............

Date ....

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Subject of Experiment

	Force F	*Perp. P*	*Angle with hor.*	*Moment*	$F \cos \alpha$	$F \sin \alpha$
Weight of body						
Attached weight						
Pulley cord						
Spring No .... Length =	$F_4 =$					

$\Sigma F \cos \alpha$ $= 0$

$\Sigma F \sin \alpha =$ $= 0$

ΣFp $= 0$

F_4 from equations =

F_4 from calibration =

F_5 from equations =

F_5 from experiment =

$\alpha =$ angle with horizontal

α_5 from equations =

a_5 from experiment = —

Form 9

NOTES

Exercise 10. The Torsional Pendulum

General Statement. (See Fig. 10.) The period is given by $t=2\pi\sqrt{\frac{Il}{\tau}}$ and if an additional mass of known moment of inertia i, be added we have $T=2\pi\sqrt{\frac{(I+i)l}{\tau}}$. Hence by getting the period without the additional mass and again with it the moment of inertia of the first body (I) can be found experimentally.

If the first body is a regular one (rectangular block) its moment of inertia may be computed from its dimensions.

Adjustments. Faces of wooden block must be horizontal and vertical.

The mirror scale used to determine the distances of the lead cylinders from the axis must be level.

Measurements. Get period by counting number of seconds required for ten half-vibrations:

1. Without lead cylinders.
2. With lead cylinders.

Get distances of cylinders from axes of suspension, also measure dimensions of wooden block.

Obtain the mass of the cylinders and block.

Calculations:

1. Calculate moment of inertia of lead cylinders about the axis of suspension. (See exercise on angular acceleration and moment of force.)

2. Calculate moment of inertia of wooden block.

(*a*) In terms of observed periods and moment of inertia of lead cylinder.

(*b*) In terms of its mass and dimensions, $\frac{1}{12}(a^2+b^2)M$.

(*c*) Compare results obtained.

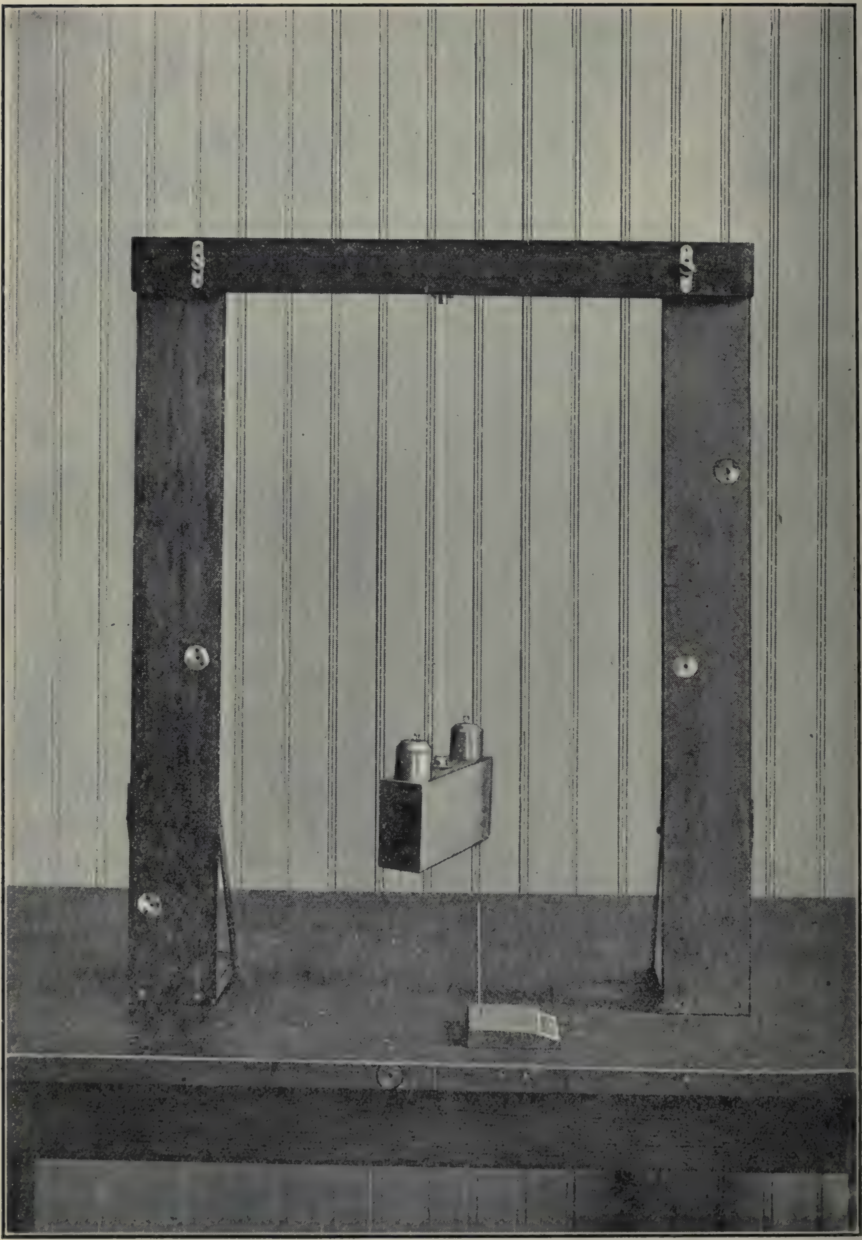

FIG. 10.

Name........................ Section........... Exercise..........

Date.....................

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Subject of Experiment

Time of ten $\frac{1}{2}$ vibrations

Without Weights	*With Weights*
Mean =	Mean =

t no weights =

T with weights =

I of lead cylinders =

I by exp. =

I from formula for rectangular block;

Mass of block =

Length of block =

Width of block =

$I = \frac{1}{12}M(a^2 + b^2) =$

Form 10

NOTES

Exercise 11. The Compound Pendulum

General Statement. (See Fig. 11.) The purpose here is to verify the relation

$$T=2\pi\sqrt{\frac{I}{mgh}}.$$

The moment of inertia can be calculated from m, h, and T.

Consult text for meaning of all symbols.

Adjustments. Axis should be horizontal.

Measurements. Find the center of gravity by balancing the rod on a knife edge.

Measure h.

Time the number of vibrations in three minutes three times.

Find m.

Calculations:

1. Calculate I from observed data.
2. Calculate I from the formula for the moment inertia of a rod.

$$I=\left(\frac{ml^2}{12}+mh^2\right).$$

3. Compare results.
4. Would you expect a very close check in this case? Why?
5. Find the length l of the equivalent pendulum (simple).
6. Repeat for several holes in the rod.
7. Plot T against h (cross-section paper).

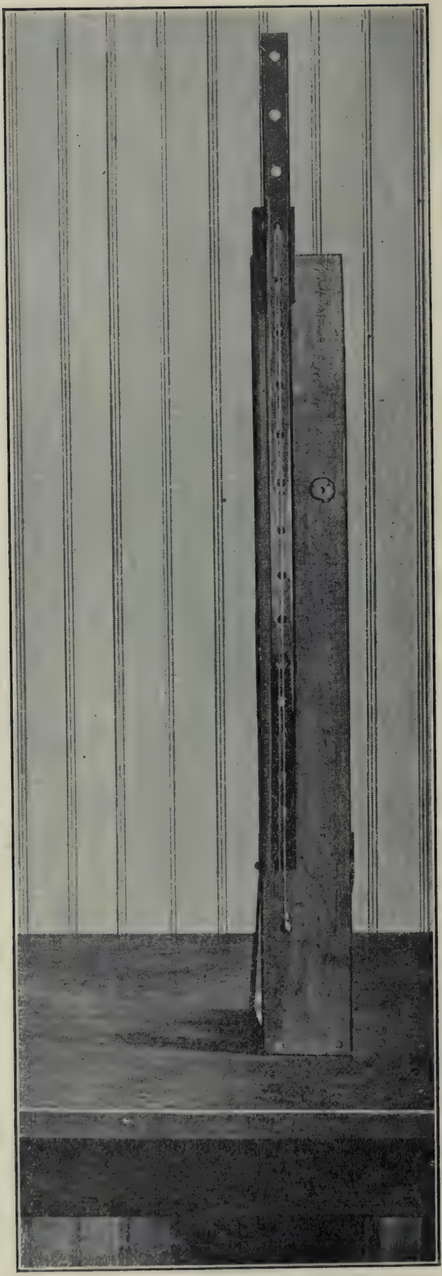

FIG. 11.

Name........................ Section............... Exercise......

Date.....................

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Subject of Experiment

No. $\frac{1}{2}$ vibrations in 3 min.
Mean =

No. $\frac{1}{2}$ vibrations in 3 min.
Mean =

No. $\frac{1}{2}$ vibrations in 3 min.
Mean =

T (time whole vibr.) =

Mass of rod =

Distance of axis from center of mass, h =

I (calculated from T, h and M) =

I (calculated from formula) =

Mass Rod =

Length Rod =

Width Rod =

Answer to 4. :

l =

Form 11

NOTES

Exercise 12. Simple Harmonic Motion

General Statement. (See Fig. 12.) The purpose of this experiment is to verify and examine the relation $T=2\pi\sqrt{-\frac{x}{a}}$, the period in S. H. M.

Consult text for meaning of all symbols.

Suspend a lead weight from a spring. The lower end of the spring in this position becomes the origin of motion. If the spring is stretched beyond zero position and released, the stretching force exceeds the weight of the lead, causing a resultant force which accelerates the lead mass toward the origin. The kinetic energy of the lead mass carries it above the origin and the spring contracts. The resultant force (the weight of the lead mass exceeding the tension on the spring) is now downward, that is, still toward the origin of the motion. Thus we can see that the force is always directed toward the origin.

Measurements. Measure several displacements, x_1, x_2, etc., from the origin and the corresponding forces, f_1, f_2, etc.

Obtain the forces f_1, f_2, etc., with a scale pan and weights.

Get the period by observing the number of vibrations in three minutes, three times.

Calculations:

1. Calculate a_1 from f_1 and m (the mass of the lead weight). Similarly for a_2, etc.

2. Find the numerical values of $\frac{x_1}{a_1}$, $\frac{x_2}{a_2}$, etc.

3. Compare the results of (2) and state your conclusion as to the character of the motion.

4. Calculate the period of the motion.

5. Enumerate the essentials of S. H. M.

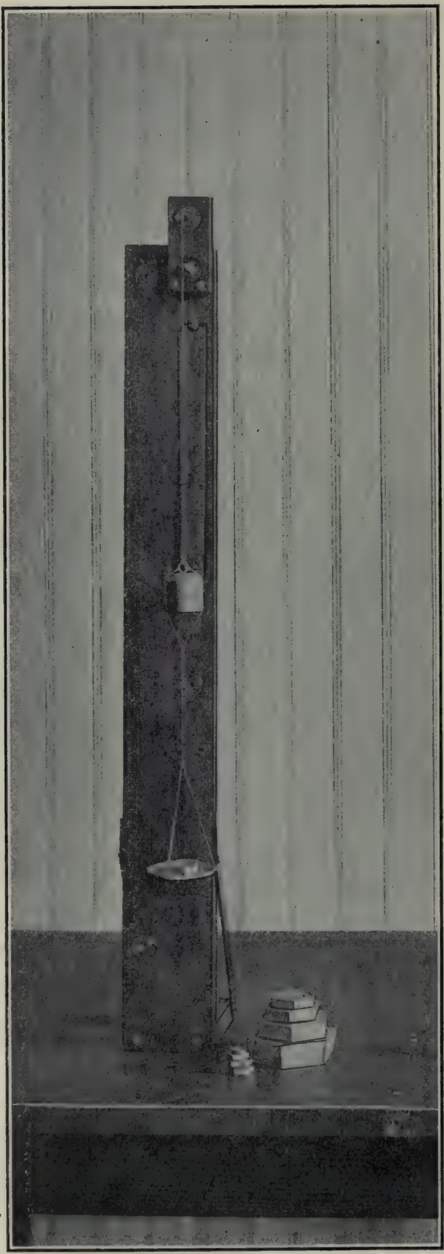

FIG. 12.

Name........................ Section.............. Exercise...........

Date ....

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(Department of Natural and Experimental Philosophy)

Subject of Experiment

Force F	Displ. x	Accel. α	$\frac{x}{\alpha}$
		Mean	

Vibrating Mass =

Calculated Period =

Vibrations in 3 min.

1. .... 2. .... 3. ....

Mean =

Observed Period =

Answer to (3):

Answer to (5):

Form 12

NOTES

Exercise 12a. Composition of Two Simple Harmonic Motions

General Statement. (See Fig. 12*a*.) If two simple harmonic motions of different periods T_1 and T_2 are combined, the resultant motion will have a variable amplitude.

If the maxima or minima of this amplitude occur n times per time unit then it will be found that $n = n_1 - n_2$ when n_1 and n_2 are the frequencies (number of vibrations per time unit of the original S. H. Motions).

NOTE. Illustrations of this occur frequently, for example, in the case of superimposed sound vibrations the combinations produce maximum and minimum sound effects called *beats.*

Adjustments. Make the period of the two vibrating weights nearly equal, that is, let the number of vibrations per minute differ anywhere from 2 to 6 (n = frequency or vibrations per minute).

Measurements:

1. Count the number of vibrations in two minutes, for each weight separately, repeating three times.

2. Attach light rod to vibrating bodies and count number of minima vibrations of the center of the rod in two minutes, three times.

Calculations. Compute the number of " beats " in two minutes.

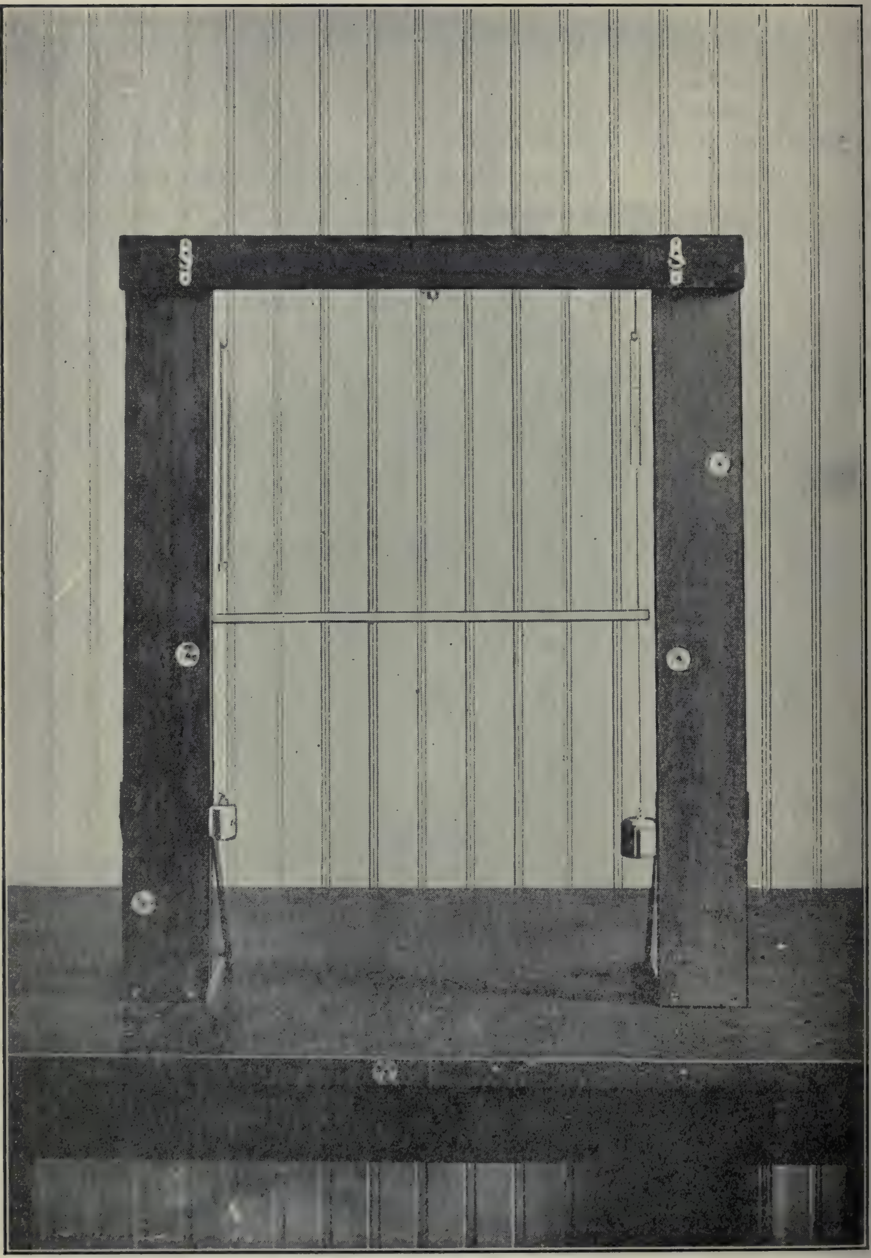

FIG. 12*a*.

Name...................... Section........... Exercise............

Date............... ..

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Subject of Experiment

n_1 = number of vibrations;
in 2 min. for weight;
1, similarly for weight 2

n = number of minima of rod in 2 min

	n_1	n_2
Mean		

$(n_1 - n_2)$ calculated =

n *observed* =

Form 12*a*

NOTES

Exercise 13. Viscosity

General Statement. (See Fig. 13.) Consult text for meaning of all symbols.

Shearing stress $=\frac{F}{A}$.

Rate of shear $=\frac{\frac{Vt}{d}}{t}=\frac{V}{d}$.

$A=2\pi rl$.

If the law of viscosity is true the shearing stress is proportional to the rate of shear or $\frac{F}{A}\propto\frac{V}{d}$, thus making $F\propto V$ and

$$\mu=\frac{\frac{F}{A}}{\frac{V}{d}}=\frac{Fd}{VA}=\frac{\text{shearing stress}}{\text{rate of shear}}.$$

If F_1 is the friction of the bearings, then $F=W-F_1$, where W is the descending weight.

Adjustments. Center rod carefully.

Fill space between cylinders carefully with a glass rod or pipette.

Adjust the pulley so that thread is horizontal.

Avoid touching the tube with the hand.

Measurements:

Get r, the radius of inner cylinder.

Get l, the inside length of the hollow cylinder.

Get R, the inner radius of the outer cylinder by filling tube with a burette three times, from volume of water.

Get time of descent with scale pan only, then with 10, 20 and 30 gms. etc.

Measure height of descent.

Calculations:

1. The velocity, V, which is the tangential velocity of the surface of the rotating cylinder, may be obtained from the height and time of descent of the weight, W. V is practically constant.

2. Plot W against V. (Cross-section paper.)

3. Deduce t_1 from the curve and plot F against V. (Cross-section paper.)

4. Take a good value of $\frac{F}{V}$ from the curve and calculate the viscosity coefficient, μ.

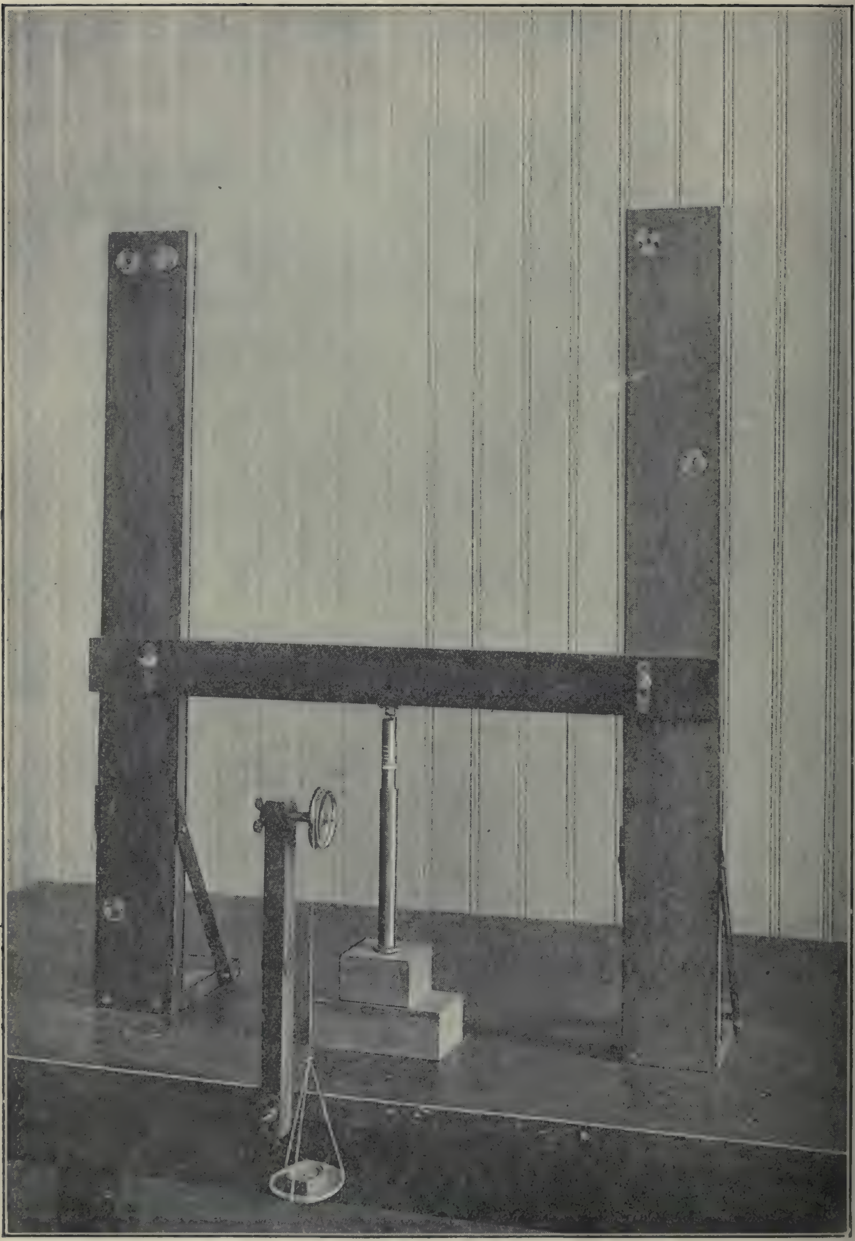

FIG. 13.

Name........................ Section........... Exercise............

Date.....................

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Subject of Experiment

Weights	Time	Velocity

Distance of descent

Friction of bearings $F^1 =$

$\frac{F}{V}$ from curve $=$

$\mu = \frac{Fd}{AV} =$

$y =$

Vol. of water $=$

$R =$

$l =$

$A =$

$d = R - r \quad =$

Form 13

NOTES

Exercise 14. Index of Refraction.

General Statement. (See reference text.)

$$\frac{V_1}{V_2} = \text{"}n\text{"} = \text{index of refraction} = \frac{\sin i}{\sin r}.$$

(a) To determine the index of refraction in a solid.

Instructions. Establish a line of sight by placing two pins vertically through a sheet of paper. Place specimen to be tested gently against one of the pins. Establish the line of sight after the ray leaves the specimen by placing two more pins in prolongation of the line of sight. Draw the outline of the specimen and remove it. Draw the full course of the deviated ray as in Fig. 14*a*.

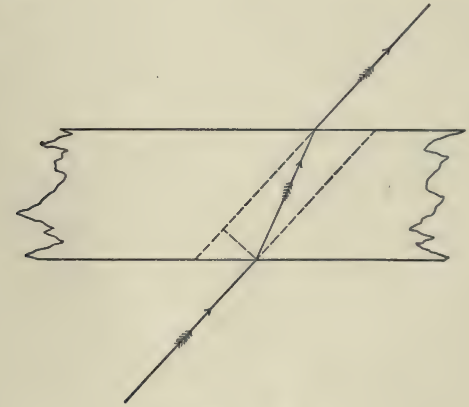

FIG. 14*a*.

Calculations:

(1) Determine the value of the ratio $\frac{V_1}{V_2} = \frac{\sin i}{\sin r}$ = (index refraction "n").

(2) Determine the velocity of light in the specimen, assuming the velocity of light in air.

(b) To obtain the index of refraction in a prism.

Given:

$$n(\text{for prism}) = \frac{\sin \frac{1}{2}(D+A)}{\sin \frac{1}{2} A}.$$

(See reference text for minimum angle of Deviation in prism $= D$.)

Instructions. Use prism as the solid specimen in 14 (*a*). (See Fig. 14*b*.) By turning the prism about a vertical axis, determine when the final ray starts to recede as you turn the prism continuously in one direction. This will give the angle D at the point of minimum deviation and the condition pictured in Fig. 14*b*.

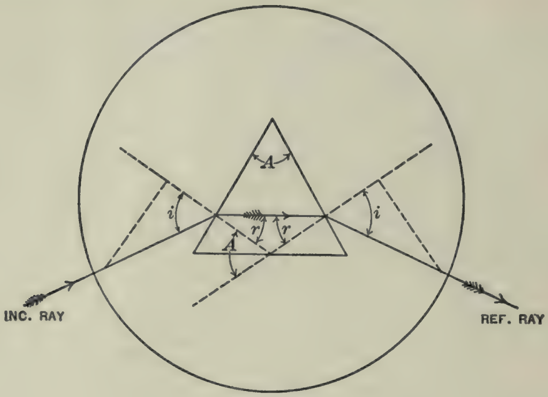

FIG. 14*b*.

Establish the line of sight and draw outline of prism as in Exercise 14 (*a*).

Calculations:

(1) Determine D.
(2) Determine A and then "n."
(3) Should it be the same as the value for "n" in 14*a*?

(c) **To determine "*a*" for a liquid.**

Instructions. Use the container filled with liquid and perform experiment exactly as in 14 (*a*).

Calculations:

(1) Value of $n = \frac{\sin i}{\sin r} = \frac{V_1}{V_2}$.

(2) Determine the velocity of light in the liquid.

Name........................ Section.............. Exercise..........

Date.....................

U. S. M. A. LABORATORY

(DEPARTMENT OF NATURAL AND EXPERIMENTAL PHILOSOPHY)

14*a*. solid

(1) $\frac{\sin i}{\sin r} = n =$ . . . . (Turn in plot on cross-section paper.)

(2) $v_2 =$ . . . .

14*b*.

(1) $D =$

(2) $\begin{cases} A = \\ \text{“} n \text{”} = \end{cases}$ (Turn in plot on cross-section paper.)

(3) Ans. to (3) =

14*c*.

$n =$

$v_2 =$

Form 14.

NOTES

Exercise 15. Mirrors and Lenses. (See reference text.)

(a) To Determine the Principal Focal Length of a Lens:

Given:

$$\frac{1}{f}=\frac{1}{u}+\frac{1}{v}=\frac{2}{r}\text{ (mirror)}, \quad . \; . \; . \; . \; . \; . \quad (1)$$

$$\frac{1}{f}=\frac{1}{u}+\frac{1}{f}=(n-1)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\text{ (lens)}. \quad . \; . \; . \; . \quad (2)$$

Instructions. *First Method.* Observe with a ground-glass screen the clearest focus of a convex lens using a distant object (200 yds. or more) and measure as carefully as possible the distance from the center of the lens to the image. Record three readings.

Calculations:

(1) Record mean of readings.

(2) Calculate the value of f obtained by solving $u+1=1$ in this experiment.

(3) Describe briefly the relative clearness and brilliancy of the image when a diaphragm is used.

Instructions. *Second Method.* To determine the focal length of a lens by use of conjugate foci: Perform Experiment 15, using for an object a lighted candle placed about 50 cm. from the lens. Record three readings of object and image distances.

Calculations:

(1) Calculate from the data recorded the value for the principle focal length of the lens.

(2) Compare with the value recorded under 1 and 2 (15).

(b) To Find an approximate Radius of Curvature of the curved surface of a Lens.

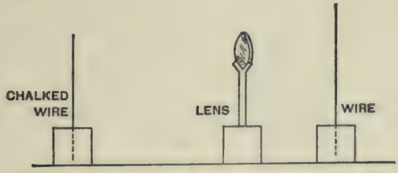

FIG. 15*a*.

Instructions. Cover one side of a convex lens, held in the mount, provided with black grease paint, or place a dark diaphragm behind it, thus converting it into a mirror.

Place a vertical wire about 25 cm. in front of the lens. On looking into the lens a virtual image of the wire is seen. This image is formed by the reflection of the rays incident on the front curved face of the lens and appears behind the lens.

Place another vertical wire in rear of the lens so it projects above and below the lens. Move the front wire until the virtual image of the front wire remains in line with the wire behind the lens as the head is moved slightly from right to left.

(Disregard the second image which appears in front of the lens. This image is the mirror image formed by the rear curved surface of the lens and appears in front of the lens.)

Calculations:

(1) Measure carefully the distance of each wire from the center of the lens when the above adjustment has been obtained. Record the readings.

(2) Using Eq. (1) determine r.

(3) Use the object and image distances obtained in 15*a* and calculate n (Eq. (2).

(4) How does this compare with n in Exercises 14*a* and 14*b*?

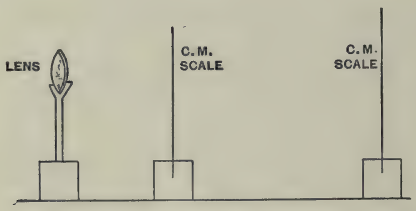

Fig. 15*b*.

(c) To Determine the "Power" of a Lens.

Instructions. Adjust a vertical C. M. scale so that it is 25 cm. from the eye; Fig. 15*b* (limit of distinct vision).

Place the lens used in the foregoing experiments in a holder. Look through the lens with the right eye (about 1 inch from the lens) and focus on a second similar vertical scale until the divisions of the first and second scales have no relative motion when the right eye is moved slightly up and down.

Observe through the lens with the right eye the two marked cm. divisions nearest the center of the field of view. These two numbered divisions on the nearest scale are seen projected on the further scale by the left eye (not looking through the lens).

Calculations:

(1) Record the readings between these projections on the further scale, and calculate the number of divisions on the further

scale seen with the naked eye, which correspond to one cm. division on the near scale seen through the lens with the right eye.

(2) Express this result as a ratio (the power of the lens).

(3) Calculate the ratio of the distances of the scales from the lens. Compare with ratio obtained in (2).

(4) Consider the scale readings as the relative size of object and image and plot to scale the lens, object, and image on cross-section paper.

Use Eq. (2), $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$ and determine the principal focal distance f. How does this compare with f obtained in first and second method, above?

(d) **To Examine the Astronomical Telescope and Field of View.** Look up astronomical telescope in reference text.

Instructions. Use the lens in 15 as in an objective and focus the convex lens on a distant object by means of a ground-glass screen. Use another short focus convex lens as an eye-piece and examine the image on the screen. Remove the screen and again examine the image. Use a third convex lens of still shorter focus and examine the same image.

Observations:

(1) How does the magnifying power vary with the focal length of the *eye-piece?*

(2) How does the extent on the field of view vary with the power of the *eye-piece?*

(3) How does the amount of light in the field of view vary with the power of the *eye-piece?*

(e) Take a convex lens of shorter focal length than the one used as an object glass, focus as before on a distant object. Use either lens used before as an eye-piece and examine the new image in the same manner as in (15*e*) and compare the effects with the image formed by the first objective lens.

Observations:

(1) How does the magnifying power vary with the focal length of the *objective lens?*

(2) How does the extent of the field of view vary with the power of the *objective lens?*

(3) How does the amount of light in the field of view vary with the power of the *objective lens?*

Name........................... Section................ Exercise........

Date..........................

U. S. M. A. LABORATORY

(DEPARTMENT OF NATURAL AND EXPERIMENTAL PHILOSOPHY)

15*a* (first method)

(1) Readings of f . . . . . , ... , ...

mean . . . .

(2) $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$ $f=$

(3) Ans to 3 =

15*a* (second method)

(1) $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$ f . . . . =

(2) Ans. to 2 =

15*b*

(1) Distance of front wire from lens . . . . . , ... , . . . ,

mean =

Distance of rear from lens . . . . . , ... , ... ,

mean =

(2) $\frac{1}{u}+\frac{1}{v}=\frac{2}{r}$, $r=$ . . . .

(3) $\frac{1}{u}+\frac{1}{v}=(n-1)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$; $(r_1=r_2)$; $n=$

(4) Ans. to 4 =

15*c* Projection of 1 C. M. on distant scale.

(1) . . . . . , ... , ... ,

mean =

(2) Power of lens =

(3) Ratio of $\frac{u}{v}=$

(4) Turn in plot on lower section paper.

15*d* Ans. to (1) =

Ans. to (2) =

Ans. to (3) =

15*e* Ans. to (1) =

Ans. to (2) =

Ans. to (3) =

Form 15.

NOTES

Exercise 16. Velocity of Sound.

(*a*) *First Method.* Given $v=n\lambda$.

Look up resonance in reference text (closed pipes).

Instructions. Insert one of the given pipes in the container filled with water and by means of the tuning fork of known frequency determine the points of adjacent maximum sounds. Record the measured distances from the open end of the pipe to the surface of the water in each case.

Calculations:

(1) Determine with the recorded data the velocity of sound in air.

(2) How would the temperature reading affect this?

(*b*) *Second Method.* Use the apparatus shown in Fig. 16.

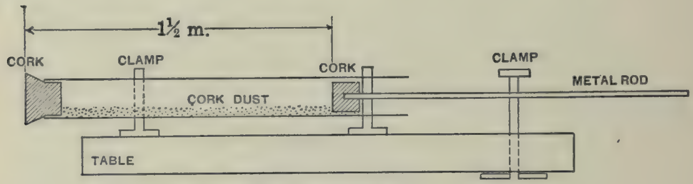

Fig. 16.

Instructions. Place the solid rod in the clamp at its center of gravity and insert the cork end carefully into the glass tube containing a small amount of cork dust. Using a resined glove stroke the rod longitudinally and move the glass tube back and forth until the particles of cork show the characteristic nodes and loops.

Measure the rod and obtain its λ from the fact that a node occurs at the clamp. Obtain the λ for the vibrations as registered by the cork particles in the glass tube.

Calculations:

(1) Using the v obtained in 16 (*a*) calculate the n from the glass tube data.

(2) Having determined n and knowing λ for the metal rod calculate the velocity of sound in the metal of which the rod is made.

(3) Since $v=\sqrt{\frac{E}{\delta}}$, determine from the above data Young's Modulus of Elasticity ("E") for metal rod.

Name........................ Section.............. Exercise...........

Date.......................

U. S. M. A. LABORATORY

(DEPARTMENT OF NATURAL AND EXPERIMENTAL PHILOSOPHY)

16(*a*)

(1) v in air =

(2) Ans. to (2)

16(*b*)

(1) Number of whole vibrations =

(2) v of sound in metal =

(3) E =

Form 16

NOTES

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